Math 20: Homework 2 Solutions

1 Problem 1

(a) In order to evaluate \( \int \arcsin x \, dx \) we will use integration by parts.

\[
\begin{array}{c|c}
  u = \arcsin x & dv = dx \\
  du = \frac{1}{\sqrt{1-x^2}} \, dx & v = x \\
\end{array}
\]

So,

\[
\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx.
\]

To evaluate \( \int \frac{x}{\sqrt{1-x^2}} \, dx \) we will use substitution with \( u = 1 - x^2 \) and \( du = -2x \, dx \). Then,

\[
\int \frac{x}{\sqrt{1-x^2}} \, dx = -\frac{1}{2} \int u^{-1/2} \, du = -\frac{1}{2} 2u^{1/2} + C = -\sqrt{1-x^2} + C.
\]

Combining these results we get

\[
\int \arcsin x \, dx = x \arcsin x + \sqrt{1-x^2} + C
\]

(b) In order to evaluate \( \int (\cos \theta)^5 \, d\theta \) we will use table entry 18 with \( n = 5 \):

\[
\int (\cos \theta)^5 \, d\theta = \frac{1}{5} \cos^4 \theta \sin \theta + \frac{4}{5} \int \cos^3 \theta \, d\theta.
\]

Now to evaluate \( \int \cos^3 \theta \, d\theta \) we again use table entry 18 with \( n = 3 \):

\[
\int \cos^3 \theta \, d\theta = \frac{1}{3} \cos^2 \theta \sin \theta + \frac{2}{3} \int \cos \theta \, d\theta = \frac{1}{3} \cos^2 \theta \sin \theta + \frac{2}{3} \sin \theta + C.
\]

Putting these results together we get that,

\[
\int (\cos \theta)^5 \, d\theta = \frac{1}{5} \cos^4 \theta \sin \theta + \frac{4}{5} \left( \frac{1}{3} \cos^2 \theta \sin \theta + \frac{2}{3} \sin \theta \right) + C = \frac{1}{5} \sin \theta \left( \cos^4 \theta + \frac{4}{3} \cos^2 \theta + \frac{8}{3} \right) + C.
\]

(c) To evaluate \( \int \frac{dt}{\sqrt{9-4t^2}} \) we will use table entry 28 with \( x = 2t \) and \( a = 3 \). Note that since \( x = 2t \) we have that \( dx = 2 \, dt \). So,

\[
\int \frac{dt}{\sqrt{9-4t^2}} = \frac{1}{2} \int \frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{2} \arcsin \frac{x}{a} + C = \frac{1}{2} \arcsin \frac{2t}{3} + C
\]
(d) First note that by multiplying by \( \frac{e^x}{e^x + e^{-x}} \) we get that \( \int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x}{e^x + 1} \). To evaluate this integral we will use substitution with \( u = e^x \). Then \( du = e^x dx \), so we have

\[
\int \frac{e^x}{e^{2x} + 1} dx = \int \frac{1}{u^2 + 1} du.
\]

This matches table entry 24 with \( a = 1 \). So, \( \int \frac{1}{u^2 + 1} du = \arctan u + C \). Putting these results together we get that

\[
\int \frac{dx}{e^x + e^{-x}} = \arctan(u) + C.
\]

(e) In order to evaluate the integral \( \int (\ln x)^2 dx \) we will use integration by parts with,

\[
u = \ln x \quad dv = \ln x \, dx \quad du = \frac{1}{x} \, dx \quad v = x \ln x - x
\]

So

\[
\int (\ln x)^2 \, dx = (\ln x)(x \ln x - x) - \int (x \ln x - x) \frac{1}{x} \, dx
\]

\[
= x \ln x(\ln x - 1) - \int (\ln x - 1) \, dx
\]

\[
= x \ln x(\ln x - 1) - (\ln x - x - x) + C
\]

\[
= \left[ x \ln x(\ln x - 1) - x(\ln x - 2) + C \right]
\]

(f) First note that \( \int \frac{dP}{P(K - P)} = -\int \frac{dP}{(P - 0)(P - K)} \). To evaluate the integral we will use table entry 26 with \( x = P \) \( a = 0 \) and \( b = K \). So,

\[
- \int \frac{dP}{(P - 0)(P - K)} = -\frac{1}{K} (\ln |x| - \ln |x - K|) + C
\]

\[
= \frac{1}{K} (\ln |x| - \ln |x - K|) + C
\]

(g) To evaluate \( \int \frac{t}{t^2 + 5} \, dt \) we will first use substitution with \( u = t^2 \) and \( du = 2tdt \). So,

\[
\int \frac{t}{t^2 + 5} \, dt = \frac{1}{2} \int \frac{1}{u^2 + 5} \, du.
\]

To evaluate \( \int \frac{1}{u^2 + 5} \, du \) we will use table entry 24 with \( a = \sqrt{5} \). So,

\[
\int \frac{1}{u^2 + 5} \, du = \frac{1}{\sqrt{5}} \arctan \frac{u}{\sqrt{5}} + C.
\]

Putting these results together we get,

\[
\int \frac{t}{t^2 + 5} \, dt = \frac{1}{2} \left( \frac{1}{\sqrt{5}} \arctan \frac{u}{\sqrt{5}} \right) + C
\]

\[
= \frac{1}{2\sqrt{5}} \arctan \frac{t^2}{\sqrt{5}} + C
\]
(h) To evaluate $\int e^{\sqrt{x}}dx$ we will use substitution with $y = \sqrt{x}$ then $dy = \frac{1}{2\sqrt{x}}dx = \frac{1}{2y}dx$. So,

$$\int e^{\sqrt{x}}dx = 2 \int ye^ydy.$$  

We can now evaluate $\int ye^ydy$ by parts.

$$\begin{array}{c|c}
u & dv \\ \hline u = y & dv = e^ydy \\ du = dy & v = e^y
\end{array}$$

So,

$$\int ye^ydy = ye^y - \int e^ydy = ye^y - e^y + C = e^y(y - 1) + C.$$

Putting these results together we get

$$\int e^{\sqrt{x}}dx = 2e^y(y - 1) + C = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C.$$  

2 Problem 2

(a) Since on the interval $-1 \leq x \leq 1$ we have that $R$ is bounded above by $y = 3 - 2x^2$ and below by $y = x^{1/3}$, the definite integral is

$$\int_{-1}^{1} (3 - 2x^2 - x^{1/3})dx$$

(b) On the interval $-1 \leq y \leq 1$, we have that $R$ is bounded on the right by $x = y^3$ and on the left by $x = -1$. On the interval $1 \leq y \leq 3$, we have that $R$ is bounded on the right by $x = \sqrt{(3-y)/2}$ and on the left by $x = -\sqrt{(3-y)/2}$. Therefore the area of $R$ is

$$\int_{-1}^{1} (y^3 - (-1))dy + \int_{1}^{3} (\sqrt{(3-y)/2} - (-\sqrt{(3-y)/2}))dy = \int_{-1}^{1} (y^3 + 1)dy + 2\int_{1}^{3} \sqrt{(3-y)/2}dy$$

(c) We will evaluate the first integral. So the area of $R$ is

$$\int_{-1}^{1} (3 - 2x^2 - x^{1/3})dx = 3x - \frac{2}{3}x^3 - \frac{3}{4}x^{4/3}\bigg|_{-1}^{1} = (3 - \frac{2}{3} - \frac{3}{4}) - (-3 + \frac{2}{3} - \frac{3}{4}) = 6 - \frac{2}{3} = \frac{14}{3}$$

(d) We cannot use the formula because $R$ is not symmetric about the $y$-axis.

3 Problem 3

(a) For a fixed $-1 \leq x \leq 1$, let $T$ be the triangle which is the cross-section of $S$ at $x$. Then the base of $T$ runs from $y = \sqrt{1-x^2}$ to $y = -\sqrt{1-x^2}$. Therefore the base has length $b = 2\sqrt{1-x^2}$, and the height is $h = 3\sqrt{1-x^2}$. So $T$ has area $\frac{1}{2}bh = 3(1-x^2)$.

This implies that the volume of $S$ is

$$\int_{-1}^{1} 3(1 - x^2).$$
(b) Evaluating the integral, we get that the volume of $S$ is

\[
\int_{-1}^{1} 3(1-x^2)dx = \int_{-1}^{1} (3-3x^2)dx \\
= 3x - x^3\bigg|_{-1}^{1} \\
= (3-1) - (-3+1) \\
= 4
\]

(c) The volume of a cone with this height and radius is $\pi$. Therefore $S$ has greater volume than the cone and $S$ is not a cone.

4 Problem 4

(a) The graph of the curve is:

![Graph of the curve](image)

(b) The region $R$ that is enclosed by the loop is bounded above by the curve $y = \sqrt{x^3 - 5x^2 + 6x}$ and is bounded below by the curve $y = -\sqrt{x^3 - 5x^2 + 6x}$. Its left and right bounds are $x = 0$ and $x = 2$. Therefore the area of $R$ is given by the integral

\[
\int_{0}^{2} (\sqrt{x^3 - 5x^2 + 6x} - (-\sqrt{x^3 - 5x^2 + 6x}))dx = 2 \int_{0}^{2} \sqrt{x^3 - 5x^2 + 6x} dx
\]

(c) For a fixed $x$ between 0 and 2, let $B$ be the square which is the cross-section of $S$ at $x$. The base of $B$ runs from $y = \sqrt{x^3 - 5x^2 + 6x}$ to $y = -\sqrt{x^3 - 5x^2 + 6x}$. Therefore the base of $B$ has length $2\sqrt{x^3 - 5x^2 + 6x}$. Since $B$ is a square this implies that the area of $B$ is $(2\sqrt{x^3 - 5x^2 + 6x})^2 = 4(x^3 - 5x^2 + 6x)$.

So the volume of $S$ is

\[
\int_{0}^{2} 4(x^3 - 5x^2 + 6x)dx = x^4 - \frac{20}{3}x^3 + 12x^2\bigg|_{0}^{2} \\
= 16 - \frac{20}{3}8 + 12(4) \\
= \frac{32}{3}
\]
5 Problem 5

(a) The implicit equation of the circle centered at \((0, R)\) with radius \(r\) is
\[ x^2 + (y - R)^2 = r^2 \]

(b) The upper boundary of \(D\) is given by \(y = f(x) = \sqrt{r^2 - x^2} + R\). The lower boundary of \(D\) is given by \(y = g(x) = -\sqrt{r^2 - x^2} + R\).

(c) Since the center of \(D\) is \((0, R)\), it extends in the \(x\)-direction between \(-r\) and \(r\). Therefore in the formula we will have \(a = -r\) and \(b = r\).

So the volume of the donut will be
\[ \int_{-r}^{r} \pi((\sqrt{r^2 - x^2} + R)^2 - (-\sqrt{r^2 - x^2} + R)^2)dx \]

(d) Before we evaluate the integral in part (c), first note that
\[
(\sqrt{r^2 - x^2} + R)^2 = (r^2 - x^2) + 2R\sqrt{r^2 - x^2} + R^2 \\
(-\sqrt{r^2 - x^2} + R)^2 = (r^2 - x^2) - 2R\sqrt{r^2 - x^2} + R^2
\]

and so
\[
(\sqrt{r^2 - x^2} + R)^2 - (-\sqrt{r^2 - x^2} + R)^2 = 4R\sqrt{r^2 - x^2}.
\]

Now we are ready to evaluate the integral.
\[
\int_{-r}^{r} \pi((\sqrt{r^2 - x^2} + R)^2 - (-\sqrt{r^2 - x^2} + R)^2)dx = \int_{-r}^{r} 4R\pi \sqrt{r^2 - x^2}dx
\]
\[
= 4R\pi \frac{j}{-r} \sqrt{r^2 - x^2}dx
\]
\[
= 4R\pi \frac{1}{2} \pi r^2
\]
\[
= \frac{2\pi^2 Rr^2}{2}
\]

6 Problem 6

First we want to set up some useful equations.

The acceleration of the stone at time \(t\) is \(a(t) = -10\). By integrating we get that the velocity at time \(t\) is \(v(t) = \int a(t)dt = -10t + C\). Since the initial upward velocity is 40 meters per second, we can solve for \(C\):
\[ 40 = v(0) = C. \]

Therefore \(v(t) = -10t + 40\). By integrating we get that the height at time \(t\) is \(h(t) = -5t^2 + 40t + C\). Since the initial height of the stone is 100 meters off the ground, we can solve for \(C\):
\[ 100 = h(0) = C. \]

Therefore the position \(h(t) = -5t^2 + 40t + 100\).

(a) The stone reaches its highest point when the velocity is 0. So when \(-10t + 40 = 0\). This happens at time \(t = 4\) seconds. 
(b) The maximum height the stone reaches is the position of the stone at time $t = 4$, so its maximum height is
\[ h(4) = -5(4^2) + 40(4) + 100 = 180 \text{ meters} \]

(c) The stone reaches the ground when the height is 0. So when $-5t^2 + 40t + 100 = 0$. This happens at $t = -2$ and $t = 10$. Since we expect a positive time, it will take 10 seconds for the stone to reach the ground.

(d) When the stone reaches the ground its velocity will be
\[ v(10) = -10(10) + 40 = -60 \text{ ms}^{-1} \]

The minus sign indicates that the stone is traveling downward.

7 Problem 7

Since the plane can accelerate from 0 to 200 miles per hour in 30 seconds. Since 30 seconds is $1/120$th of an hour, this means that the plane’s acceleration is $\frac{200 \text{ mi/hr}}{1/120 \text{ hr}} = 24000 \text{ mi/hr}^2$. Since acceleration is constant this makes the formula for accelerate $a(t) = 24000$. Integrating we get that the plane’s velocity at time $t$ is $v(t) = 24000t + C$. Since the velocity at time $t = 0$ is 0, we have that $C = 0$ and so $v(t) = 24000t$. Integrating again, we get that the position of the plane on the runway at time $t$ is $p(t) = 12000t^2 + C$. Since the plane is initially 0 miles down the runway, $C = 0$ and so $p(t) = 12000t^2$. Since the plane will reach its takeoff velocity at time $t = 1/120$ hours, the runway needs to be
\[ p(1/120) = 12000(1/120)^2 = \frac{5}{6} \text{ miles} \]

8 Problem 8

(a) $y = f(t) = \frac{1}{t}$ and $y' = f'(t) = -\frac{1}{t^2} = -\left(\frac{1}{t}\right)^2 = -y^2$. Therefore $f(t)$ satisfies equation \[ IV \]. The initial value at 0 is not defined. Instead you should choose an initial value at a different point. For example, $y_1 = f(1) = 1$.

(b) $y = f(t) = e^{5t}$. Then $y' = f'(t) = 5e^{5t} = 5y$. Therefore $f(t)$ satisfies equation \[ III \]. The initial value is $y_0 = f(0) = 1$.

(c) $y = f(t) = \frac{3}{2}t^2 + t + 100$. Then $y' = f'(t) = 3t + 1$. Therefore $f(t)$ satisfies equation \[ II \]. The initial value is $y_0 = f(0) = 0$.

(d) $y = f(t) = \cos(4t)$. Then $y' = f'(t) = -4\sin(4t)$ and $y'' = f''(t) = -16\cos(4t) = -16y$. Therefore $f(t)$ satisfies equation \[ I \]. The initial value is $y_0 = f(0) = 1$.

(e) $y = f(t) = \frac{3}{2}t^2 + t$. Then $y' = f'(t) = 3t + 1$. Therefore $f(t)$ satisfies equation \[ II \]. The initial value is $y_0 = f(0) = 0$.

(f) $y = f(t) = 5\sin(4t)$. Then $y' = f'(t) = 20\cos(4t)$ and
\[ y'' = f''(t) = -80\sin(4t) = -16(5\sin(4t)) = -16y. \]

Therefore $f(t)$ satisfies equation \[ I \]. The initial value is $y_0 = f(0) = 0$. 

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