Problems

1. Solve the differential equation \( y' = e^y \sin t \) for the solution satisfying \( y(0) = 0 \).

2. Leaves fall onto a forest floor at a constant rate of 12 grams per square centimeter per day. The leaves decay at a constant rate of 40% per day.
   
   a. Complete the differential equation that models \( L(t) \), the amount of leaves on the forest floor after \( t \) days:

   \[
   \frac{dL}{dt} = \text{__________} - \text{__________} \cdot L
   \]

   b. Draw the phase diagram for this autonomous differential equation. Determine whether each equilibrium is stable, unstable, or metastable.

   c. If \( L_0 \) (the initial amount of leaves) is 30 g/cm\(^2\), what is \( \lim_{t \to \infty} L(t) \) equal to?

   *Hint: Don’t solve the diffeq in (a) unless you absolutely have to!*

   d. If \( L_0 \) (the initial amount of leaves) is 400 g/cm\(^2\), what is \( \lim_{t \to \infty} L(t) \) equal to?

3. A circular oil spill spreads at a rate given by the differential equation

   \[
   \frac{dr}{dt} = k \frac{1}{r}
   \]

   where \( r \) is the radius of the spill. If the radius of the spill is 400 feet after 16 hours, what is the value of \( k \)? (Assume the radius of the spill is 0 feet, initially). Include units.

4. a. Draw the phase diagram for the differential equation \( y' = y(3 - y)^2(2y + 4) \) and classify the equilibria as stable, unstable, or metastable.

   b. Suppose that \( y = f(t) \) is a solution to the differential equation \( y' = \sin y \cos y \). If \( f(0) = \pi/4 \), then what is the value of \( \lim_{t \to \infty} f(t) \)?

5. Consider a pyramid with (i) a hexagonal base of area 18 and (ii) a height of 6.
   
   To compute the volume of this pyramid, we let \( t \) vary from 0 (the apex of the pyramid) to 6 (the center of the base of the pyramid), and integrate the areas of the cross-sections through \( t \): \( V = \int_0^6 A(t) \, dt \).

   *Note: You do not need to know how to compute the areas of hexagons to complete this problem.*

   a. Compute \( A(0) \) and \( A(6) \).

   b. A hexagon is a two-dimensional figure. If a hexagon is scaled by half in both the \( x \)-dimension and the \( y \)-dimension, then how will the original and scaled hexagons’ areas compare?

   Using this, compute \( A(3) \), the area of the cross-section half-way down (or up!) the pyramid.

   c. What is \( A(2) \)? (This is one third of the way down the pyramid.)
d. Find \( A(t) \), set up an integral for the volume of the pyramid, and then compute the volume of the pyramid.

6. Let \( R \) be the region of the \( xy \)-plane bounded above by \( y = \frac{1}{2}x \) and \( y = 2^{-x} \), and bounded below by the \( x \)-axis and \( y = \frac{\sqrt{x-1}}{4} \). The region is bounded on the left by \( x = 0 \) and on the right by \( x = 7 \). Let \( S \) be the solid obtained by revolving \( R \) around the \( x \)-axis.

   a. (Optional.) Sketch a graph and verify that the upper and lower boundaries of the region change at \( x = 1 \) and that \( x = 2 \) is the right boundary of the region.

   b. Set up an integral or sum of integrals for the volume of \( S \).

   \[ \text{You will need to know the formula for volumes of solids of revolution for the exam.} \]

   c. Evaluate the integral.

7. Solve the following initial value problems:

   a. \( y' = -5\sin(3t) + 2\cos(4t) \) with \( y(0) = 0 \).

   b. \( y' = \frac{y}{\ln y} \) with \( y(0) = 1 \). \( \text{This one has two possible solutions. Either is enough (or both!).} \)

   c. \( y' = e^{-y}(1 + 2t + 3t^2) \) with \( y(0) = 1 \).

8. Match the graphs below with the following situations:

   a. The concentration of tree pollen over the course of a year.

   b. The speed of a car traveling at uniform speed and then breaking uniformly.

   c. The mass of carbon-14 in an archaeological sample.

   d. The population of a new species introduced to a tropical island.

   e. The temperature of a metal ingot placed in a furnace, and then removed.

   f. One of the graphs above is \( y = P(t) \) where \( P \) is a solution to a differential equation of the form \( P' = kP \). Which? and what is this differential equation called?

   g. One of the graphs above is \( y = P(t) \) where \( P \) is a solution to a differential equation of the form \( P' = rP(1 - \frac{P}{K}) \). Which? and what is this differential equation called?

9. Water leaks from a tank through a small hole at a rate proportional to the square root of the volume of water remaining.

   a. Write down a differential equation that models the volume of water remaining in the tank.

   b. Suppose that the tank initially contains 225 liters of water. Solve for \( V(t) \), the volume of water in the tank at time \( t \).
c. How long until the tank is empty? (Your answer will still have \( k \) in it, but should no longer have \( C \).)

10. A spherical snowball melts at a rate proportional to its surface area.
   a. Write down a differential equation that models the volume of the snowball. Your answer may include, in addition to \( V \), \( t \), and \( k \) (the constant of proportionality), \( A \), a second dependent variable (the surface area of the snowball).
   b. Using the formulas \( V = \frac{4}{3}\pi r^3 \) and \( A = 4\pi r^2 \), solve for \( A \) in terms of \( V \), and use this to rewrite the equation from (a), eliminating \( A \).
   c. Your answer in (b) should look like \( \frac{dV}{dt} = KV^p \) for some (cumbersome) constant \( K \) and some power \( p \). Leaving \( K \) as \( K \) and replacing \( p \) with the correct value, solve for \( V(t) \), the particular solution to the differential equation satisfying \( V(0) = V_0 \).

11. Let \( E \) be an ellipse with \( x \)-radius \( a \) and \( y \)-radius \( b \). \( E \) can be graphed using the implicit equation
   \[
   \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
   \]
   a. Derive a formula for the area of \( E \) (in terms of \( a, b \)) using integration.
   b. Let \( S \) be the solid whose base is \( E \) and whose cross-sections perpendicular to the \( x \)-axis are squares. Find the volume of \( S \) (in terms of \( a, b \)).
   c. Suppose now that we take \( a = 3 \) and \( b = 2 \). Let \( S \) be as in (b). Let \( S' \) be the solid whose base is \( E \) and whose cross-sections perpendicular to the \( y \)-axis(!) are squares. How do the volumes of \( S \) and \( S' \) compare?

12. The radius of the star S-8472 over its lifespan satisfies the autonomous differential equation
   \[
   \frac{dr}{dt} = r(1 - r)^2(3 - r)
   \]
   where \( r \) is measured in millions of kilometers and time is measured in thousands of years.
   a. Partario wants to solve for the general solution \( r(t) \). Here’s how he starts:
   \[
   \frac{dr}{r(1 - r)^2(3 - r)} = dt
   \]
   \[
   \int \frac{dr}{r(1 - r)^2(3 - r)} = \int dt
   \]
   \[
   \ln |r(1 - r)^2(3 - r)| = t + C
   \]
   \[
   |r(1 - r)^2(3 - r)| = e^C \cdot e^t
   \]
   \[
   r(1 - r)^2(3 - r) = K e^t \quad \text{where } K \text{ is a constant } (\pm e^C)
   \]
   He’s not done solving for \( r \), but explain why he’s already wrong.
   b. Suppose the radius of S-8472 at \( t = 0 \) is 700,000 kilometers. What do you expect to be true about S-8472’s radius as time goes on?
   c. Suppose the radius of S-8472 at \( t = 0 \) is 1,000,000 kilometers but a couple million years from now, it collides with and engulfs another star. Without knowing how big this star is, can you predict what will happen to S-8472 in the long term?