Math 20, Spring 2019 — Schaeffer
Stanford University

Homework 6, due 6/05 at 10 AM on Gradescope

Information and instructions

This assignment is a gentle introduction/exploration of the last topic in the course, parametric equations. The relevant lectures are those of 5/24 (Lecture 22) and 5/29 (Lecture 23), and the assignment is due at 10 AM on the last day of class, 6/05—Homework 7 is due the same day and will be published on the course website somewhat later.

The relevant textbook section is Section 4.8. You may notice that this is before integration is covered in the textbook, and hence there are no actual integrals in this assignment. Homework 7 will cover some interactions between parametric equations and integration (arc length, specifically, in Section 8.1).

While parametric equations may seem like a random topic to cover in Math 20, it generalizes what you’ve learned in Math 19/20 about 1-dimensional motion to 2 (or more) dimensions. Also, of the topics covered in Math 20, parametric equations are the most directly relevant to the material in Math 51.

Instructions: Read Section 4.8 and complete all problems.

Graphing parametric equations in Desmos

Desmos [https://www.desmos.com/calculator] can graph parametric equations for you. There are two ways to go about doing this:

• To graph the path traced by a particle moving with coordinate functions \((x(t), y(t))\) and \(a \leq t \leq b\), simply type in \((___, ___)\) where the two blanks are expressions in \(t\).

• For an animation of the particle moving in the \(xy\)-plane according to those equations, write the same \((___, ___)\) substituting a different variable for \(t\) (such as \(s\)). Desmos will prompt you to add a “slider for \(s\)” and once you do, you can set the bounds for \(s\), and then press the play button (to the left of the expression in the desktop interface) to watch the particle move back and forth along its path.

For the sake of clarity, if you’re studying a particular parametric equation, it’s good to do both of the above. Doing so will show you both the path and the particle moving along it.

Problems

1. Describe the curves traced by particles moving in the \(xy\)-plane according to the parametric equations in (a–h). You do not have to draw the graphs yourself, but Desmos is still extremely helpful here.

   For those marked with *, also solve for the \(y = \text{form of the curve}\).

   For those marked with **, the path of the particle lies along the unit circle \(x^2 + y^2 = 1\). In these cases, explain also how the particle moves (where it starts, where it ends, whether it travels clockwise or counterclockwise, and how many times it travels around the circle).

   a.* \((t^2, t)\) with \(-4 \leq t \leq 0\)
   b.** \((\sin t, \cos t)\) with \(0 \leq t \leq 6\pi\)
   c.* \((5t + 3, 2t - 6)\) with \(-2 \leq t \leq 3\)
   d.* \((t, \sin t)\) with \(-\pi \leq t \leq \pi\)
6. The parametric equations

\[
\left( \frac{t \sin t}{10}, \frac{t \cos t}{10} \right) \quad \text{with } 0 \leq t \leq 20
\]

f.* \((\cos t)^2, (\sin t)^2\) \quad \text{with } 0 \leq t \leq \pi

g. \((7 \cos t - 2 \cos \left( \frac{7t}{2} \right), 7 \sin t + 2 \sin \left( \frac{7t}{2} \right)) \quad \text{with } 0 \leq t \leq 4\pi

h.* \((-\cos t, \sin t) \quad \text{with } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\)

2. a. Use trig identities to explain why the curve traced by \((\sec(t), \tan(t))\) with \(-\frac{\pi}{3} \leq t \leq \frac{\pi}{3}\) lies along the hyperbola \(x^2 - y^2 = 1\). What are the endpoints of the hyperbolic arc that is traced?

b. The parametric equations \((\sin(t), \cos(2t))\) with \(0 \leq t \leq 2\pi\) trace a parabolic arc. Which parabola (in \(y = \) form) does this path lie along, and what are the endpoints of the arc it draws? Again, you’ll want to consult a table of trig identities and find one relating \(\sin t\) and \(\cos(2t)\).

3. Suppose some curve is traced by \((x(t), y(t))\) with \(a \leq t \leq b\). We will now investigate what happens to the curve when you modify the parametric equations that drew it.

If you want an example to experiment with in Desmos, use \((t, \sin t)\) with \(-\pi \leq t \leq 2\pi\).

a. Describe the curve traced by \((2x(t), y(t))\) with \(a \leq t \leq b\).

b. Describe the curve traced by \((x(t), 3y(t))\) with \(a \leq t \leq b\).

c. Describe the curve traced by \((x(t) + 1, y(t))\) with \(a \leq t \leq b\).

d. Describe the curve traced by \((x(t), y(t) - 4)\) with \(a \leq t \leq b\).

e. Describe the curve traced by \((-x(t), y(t))\) with \(a \leq t \leq b\).

f. Describe the curve traced by \((x(t), -y(t))\) with \(a \leq t \leq b\).

g. Describe the curve traced by \((y(t), x(t))\) with \(a \leq t \leq b\).

4. The parametric equations \((\cos t, \sin t)\) with \(0 \leq t \leq 2\pi\) trace the unit circle \(x^2 + y^2 = 1\) starting and ending at \((1, 0)\) and traveling counterclockwise. Using this information, plus what you observed in the last problem, find parametric equations that graph an ellipse with \(x\)-radius \(r\), \(y\)-radius \(s\), and center at the point \((a, b)\). Hint: Stretch, then shift, in that order.

5. a. Graph the curves traced by \((2t, t + 1), (2t^2, t^2 + 1)\) and \((2t^3, t^3 + 1)\) with \(-1 \leq t \leq 1\).

b. Explain why all of these parametric equations trace a segment of the line \(y = \frac{1}{2}x + 1\).

c. Explain why \((2t^2, t^2 + 1)\) only traces half the line segment that \((2t, t + 1)\) and \((2t^3, t^3 + 1)\) do.

d. Even though the paths traced by \((2t, t + 1)\) and \((2t^3, t^3 + 1)\) for \(-1 \leq t \leq 1\) are identical, explain how the movement of the corresponding particles along these paths are different.

Verify your observation mathematically by computing the speed functions of these particles, graphing speed versus time as \(t\) varies from \(-1\) to \(1\).

6. The parametric equations

\[
(16(\sin t)^3, 13 \cos(t) - 5 \cos(2t) - 2 \cos(3t) - \cos(4t)) \quad \text{with } 0 \leq t \leq 2\pi
\]

trace a heart-shaped curve.

a. Find all points (in \((x, y)\) form) on this curve where \(x'(t) = 0\).

b. Does the particle \(\text{stop at any point?}\)