Midterm 1

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Failure to follow the instructions below is a breach of the Stanford Honor Code:

- You may not use or consult any book or notes during the exam.*
- You may not use a calculator or the calculator function on any electronic device during the exam.*
- You may not access any internet-capable electronic device during the exam,* including smartphones and smartwatches, for any reason. These devices must be switched to “airplane mode” and disconnected from all wireless networks (both cellular and wifi) during the exam*.
- You must sit in your assigned seat.
- You may not communicate with anyone other than the course staff during the exam,* or look at anyone else’s solutions.

*“During the exam” is defined as: After you start the exam, and before you turn in the exam and leave the testing site.

- You have 50 minutes to complete this exam. If the course staff must ask you to stop writing or to turn in your exam more than once after time is called, you may receive a score of zero.

I understand and accept these instructions. All smart devices on my person are in airplane mode and disconnected from all wireless networks.

Signature: _______________________________________________________

Remember to show your work and justify your answer if required (additional tips are on the next page). Present all solutions in as organized a manner as possible. GOOD LUCK!
Here are some tips:

- If you have time, it’s always a good idea to **check your work**.
- If you get the wrong answer for an integral but **show your work**, chances are good that we can award you partial credit.
- **DO NOT** attempt to estimate any of your answers as decimals. For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is **exact**.
- The boxes at the end of each topic are for grading purposes only. **Do not touch or look at these boxes.** Pretend they are not there.
- The last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

Integration table entries you might need:

I. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C$

II. $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C$

IIIa. $\int \sqrt{u^2 + a^2} \, du = \frac{1}{2} u \sqrt{u^2 + a^2} + \frac{a^2}{2} \ln \left| u + \sqrt{u^2 + a^2} \right| + C$

IIIb. $\int \sqrt{u^2 - a^2} \, du = \frac{1}{2} u \sqrt{u^2 - a^2} - \frac{a^2}{2} \ln \left| u + \sqrt{u^2 - a^2} \right| + C$

IV. $\int \sqrt{a^2 - u^2} \, du = \frac{1}{2} u \sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin \left( \frac{u}{a} \right) + C$

Trigonometric identities you might need:

\[ \sin^2 x + \cos^2 x = 1 \quad \sin^2 x = 1 - \cos^2 x \quad \cos^2 x = 1 - \sin^2 x \]

\[ \sin^2 x = \frac{1 - \cos(2x)}{2} \quad \cos^2 x = \frac{1 + \cos(2x)}{2} \]

Above, $\sin^nx$ means $(\sin x)^n$. 
1. a. Which of the following is/are true statements about the definite integral?

Circle all true statements.

i. If \( f(x) \) and \( g(x) \) are continuous, then \( \int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \).

ii. If \( f(x) \) is a differentiable function, then \( \int_a^b f'(x) \, dx = f(b) - f(a) \).

iii. If \( f(x) \) is a differentiable function, then \( \int_a^b f'(x) \, dx = f(x) + C \).

iv. If \( f(x) \) is a continuous function, then \( \int_a^b f(x) \, dx = \left| \int_a^b f(x) \, dx \right|^2 \).

v. If \( a < b < c \) then \( \int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx \).

vi. If \( n \) is an odd positive integer, then \( \int_{-1}^1 x^n \, dx = 0 \).

vii. None of the above.

True statements: (i) is linearity of the integral (“vertical stacking”). (ii) is the FTC. (v) is additivity of the integral (“horizontal stacking”). (vi) follows either from geometry (an odd power of \( x \) has a graph that has rotational symmetry around the origin, so \( \int_{-1}^1 \) and \( \int_0^1 \) will cancel each other out) or the power rule for integration.

False statements: (iii) A definite integral is always equal to a number but here \( f(x) \) is a function and the \(+C\) indicates that the right hand side is in the form seen for indefinite integrals. The formula \( \int f'(x) \, dx = f(x) + C \) is correct, but the integral on the left is now indefinite. (iv) is false, as we’ve pointed out several times—integration does not “cooperate” with multiplication. For instance, if \( f(x) = x \), then \( \int_0^1 [f(x)]^2 \, dx = \int_0^1 x^2 \, dx = 1/3 \) but \( \left[ \int_0^1 f(x) \, dx \right]^2 = \left[ \int_0^1 x \, dx \right]^2 = (1/2)^2 = 1/4 \).

b. Which of i.–vi. in part (a) is the (First) Fundamental Theorem of Calculus?

Statement (ii) above is the FTC.

2. a. Which of the following is an antiderivative of \( 2^x \)? Circle the correct answer.

\[
2^x \quad (\ln 2) \cdot 2^x \quad \frac{2^x + 1}{x + 1} \quad \text{None}
\]

The derivatives of the above functions are \((\ln 2) \cdot 2^x\), \((\ln 2)^2 \cdot 2^x\), and some quotient rule mess. None are \( 2^x \). We in fact have \( \int 2^x \, dx = \frac{2^x}{\ln 2} + C \) (this is one of the integral formulas on the study guide), so \( \frac{2^x}{\ln 2} \) would have been correct.

The third option above “looks like” the power rule, but remember, the power rule applies to \( \int x^p \, dx \) where the base \( x \) is variable and the exponent \( p \) is constant. \( 2^x \) is not a power function, it is an exponential function (the base is fixed and the exponent is variable).

b. Which of the following is an antiderivative of \( e^{x^3} \)? Circle the correct answer.

\[
\frac{e^{x^3}}{3x^2} \quad 3x^2 e^{x^3} \quad e^{x^3} \quad \text{None}
\]

Again, none of these are the antiderivative (can be checked by differentiation). The first answer is the result of applying the “baby” substitution rule incorrectly, to a composite function where the inner function (in this case \( x^3 \)) is NOT linear (this was 10bc on HW2). The second answer is the derivative of \( e^{x^3} \). The third answer is the result of applying the substitution \( u = x^3 \) but not accounting for the difference between \( dx \) and \( du = 3x^2 \, dx \) (this was 10a on HW2).

3. Integration by parts is an integration technique that is based on which derivative rule?

Integration by parts is based on the product rule.

Integration by substitution is an integration technique that is based on which derivative rule?

Integration by substitution is based on the chain rule.

Both of these were explicitly mentioned in the study guide.
4. Suppose a tank of water is filling up at a rate of \( R(t) \) cubic meters per hour. You may assume that \( R(t) \) is always positive and that \( t \) is measured in hours. Express the following quantity as a definite integral: The amount of water gained by the tank over the three hours beginning at \( t = 2 \).

\[
\int_{2}^{5} R(t) \, dt
\]

5. Below is graphed \( y = f(t) \).

Let \( F(x) = \int_{2}^{x} f(t) \, dt \). **Caution:** The graph above is of \( y = f(t) \), **not** of \( y = F(x) \).

- **a.** Is \( F(0) \) positive, negative, or zero?
  
  By plugging in, \( F(0) = \int_{2}^{0} f(t) \, dt = -\int_{0}^{2} f(t) \, dt \).
  
  The integral \( \int_{0}^{2} f(t) \, dt \) is positive, so \( F(0) \) is **negative**.

- **b.** Evaluate \( F'(10) \).
  
  \( F'(x) = f(x) \), so \( F'(10) = f(10) = 6 \) (the value indicated by the graph).
The problems on this page refer to the functions \( f(t) \) and \( F(x) \) from the previous page.

c. At which point(s) (x-value(s)) in the interval \( 0 \leq x \leq 25 \) is \( F'(x) = 0 \)?

\( F'(x) = f(x) \), so we just need to find where \( f(x) = 0 \), and the only such point is \( x = 20 \).

d. At what point (x-value) in the interval \( 0 \leq x \leq 25 \) is \( F(x) \) maximized?

From differential calculus we know that the maximum must occur either at an endpoint or a critical point of \( F(x) \). The points to check are \( x = 0 \) (endpoint), \( x = 20 \) (critical point), and \( x = 25 \) (endpoint).

\( F(0) \) is negative by part (a). From the graph of \( y = f(t) \), it is clear that \( F(25) < F(20) \). Thus, the maximum must occur at \( x = 20 \).

This can also be reasoned/explained graphically. As \( x \) moves from left to right, starting at \( x = 5 \), the integral that defines \( F(x) \) gains positive area until \( x = 20 \), after which it gains negative area since \( y = f(t) \) crosses to being below the \( x \)-axis. The maximum therefore occurs at \( x = 20 \).

e. Now, let \( G(x) = \int_{0}^{x} f(t) \, dt \). Briefly explain why \( F(x) - G(x) \) must be a constant.

This problem is on HW3.
In problems 6–10, evaluate the indefinite integral, showing your work. If you use an integration technique or a table entry, be sure to indicate that clearly. Note that there are integration table entries and trigonometric identities on the second page of the exam. These may prove helpful!

For each of 6–10, draw a box around your final answers.

6A. \( \int [\cos(3x) - 2\sin(5x)] \, dx \)

\[
\int [\cos(3x) - 2\sin(5x)] \, dx = \int \cos(3x) \, dx - 2 \int \sin(5x) \, dx \\
= \frac{\sin(3x)}{3} + \frac{2\cos(5x)}{5} + C
\]

7A. \( \int x^5 \ln x \, dx \)

The integrand is a product and also contains a logarithm, so this calls for integration by parts.

We choose \( u = \ln x \) and \( dv = x^5 \, dx \), so \( du = \frac{1}{x} \, dx \) and \( v = \frac{1}{6}x^6 \). Using IbP:

\[
\int x^5 \ln x \, dx = \frac{x^6 \ln x}{6} - \frac{1}{6} \int x^6 \cdot \frac{1}{x} \, dx \\
= \frac{x^6 \ln x}{6} - \frac{1}{6} \int x^5 \, dx \\
= \frac{x^6 \ln x}{6} - \frac{x^6}{36} + C
\]

8A. \( \int \sqrt{x - 9} \, dx \)

We substitute \( u = x - 9 \) and \( du = dx \).

\[
\int \sqrt{x - 9} \, dx = \int (x - 9)^{1/2} \, dx \\
= \int u^{1/2} \, du \\
= \frac{2u^{3/2}}{3} + C \\
= \frac{2(x - 9)^{3/2}}{3} + C
\]

9A. \( \int x^2 e^{4x^3 + 1} \, dx \)

We substitute \( u = 4x^3 + 1 \) and \( du = 12x^2 \, dx \), so \( x^2 \, dx = \frac{1}{12} \, du \).

\[
\int x^2 e^{4x^3 + 1} \, dx = \frac{1}{12} \int e^u \, du \\
= \frac{e^u}{12} + C \\
= \frac{e^{4x^3 + 1}}{12} + C
\]
10A. \[ \int \sqrt{4x^2 - 4x + 5} \, dx \] — Hint: \((2x - 1)^2 = \)??

The hint allows us to complete the square: \(4x^2 - 4x + 5 = (2x - 1)^2 + 2^2\). We therefore apply table entry IIIa with \(u = 2x - 1\) and \(a = 2\), remembering that \(du = 2 \, dx\) and therefore \(dx = \frac{1}{2} \, du\):

\[
\int \sqrt{4x^2 - 4x + 5} \, dx = \frac{1}{2} \int \sqrt{u^2 + a^2} \, du \\
= \frac{1}{2} \left( \frac{2x - 1}{2} \right) \sqrt{(2x - 1)^2 + 2^2} \ln \left| (2x - 1) + \sqrt{(2x - 1)^2 + 2^2} \right| + C \\
= \frac{(2x - 1)\sqrt{4x^2 - 4x + 5}}{4} + \ln \left| (2x - 1) + \sqrt{4x^2 - 4x + 5} \right| + C
\]

6B. \[ \int [2 \cos(5x) - \sin(7x)] \, dx \]

Similar to (6A): \[ \int [2 \cos(5x) - \sin(7x)] \, dx = \frac{2\sin(5x)}{5} + \frac{\cos(7x)}{7} + C \]

7B. \[ \int x^5 \ln x \, dx \]

Similar to (7A): \[ \int x^5 \ln x \, dx = \frac{x^5 \ln x}{5} - \frac{x^5}{25} + C \]

8B. \[ \int \sqrt{x - 4} \, dx \]

Similar to (8A): \[ \int \sqrt{x - 4} \, dx = \frac{2(x - 4)^{3/2}}{3} + C \]

9B. \[ \int xe^{-x^2} \, dx \]

Similar to (9A), substituting \(u = -x^2\): \[ \int xe^{-x^2} \, dx = -\frac{e^{-x^2}}{2} + C \]

10B. \[ \int \sqrt{4x^2 + 4x} \, dx \] — Hint: \((2x + 1)^2 = \)??

Similar to (10A), but \(4x^2 + 4x = (2x + 1)^2 - 1\), so we use IIIb with \(u = 2x + 1\) and \(a = 1\):

\[
\int \sqrt{4x^2 + 4x} \, dx = \frac{(2x + 1)\sqrt{4x^2 + 4x}}{4} - \frac{1}{4} \ln \left| (2x + 1) + \sqrt{4x^2 + 4x} \right| + C
\]