1. Clover, our dog friend, has finally completed building his rocket ship, and will now return to his home planet. His ship’s acceleration is given by

\[ a(t) = 10000e^{-2t} \] (in miles per second per second)

a. Assuming Clover’s initial position and initial velocity are both zero, solve for \( p(t) \), the function that gives Clover’s position (in miles) \( t \) seconds after lift-off. 

*Draw a box around your final answer.*

We have

\[ v(t) = \int a(t) \, dt = \int 10000e^{-2t} \, dt = -5000e^{-2t} + C_1 \]

Plugging in \( t = 0 \) above and using \( v(0) = 0 \), we see that \( C_1 = 5000 \). Next,

\[ p(t) = \int v(t) \, dt = \int (-5000e^{-2t} + 5000) \, dt = 2500e^{-2t} + 5000t + C_2 \]

Plugging in \( t = 0 \) above and using \( p(0) = 0 \), we see that \( C_2 = -2500 \). So:

\[ p(t) = 2500e^{-2t} + 5000t - 2500 \]

is the final answer.

b. If Clover’s home planet is 7 million miles away from Earth, briefly explain how you would figure out when he arrives. (Do not attempt to solve this problem.)

Solve \( p(t) = 7000000 \) for \( t \). If you’re curious, the answer is \( \approx 1400.5 \) seconds. So Clover will arrive in 23 minutes. Better than my commute! :P

2. Solve the differential equation \( y' = \frac{\sin t}{e^y} \) for the general solution \( y(t) \).

We have

\[ y'(t)e^{y(t)} = \sin t \]

Integrating both sides by \( t \),

\[ \int y'(t)e^{y(t)} \, dt = \int \sin t \, dt \]

\[ e^{y(t)} = -\cos t + C \]

\[ y(t) = \ln(-\cos t + C) \]

(where for the left integral we use \( u = y(t) \) and \( du = y'(t) \, dt \)).

3. For each of the following situations, circle the differential equation that we studied that is relevant in that particular situation.

a. Neptunium-235 decays into Proactinium-231 and Uranium-235 with a half-life of 396.1 days. You want a function \( P(t) \) that expresses how much Neptunium-235 is present in a sample after \( t \) days.

\[ P' = rP \text{ with } r > 0 \quad P' = rP \text{ with } r < 0 \quad P' = rP(1 - P/K) \]
b. You are depositing money in a savings account that continuously compounds (positive) interest. You want a function \( P(t) \) that expresses how much money will be in the account after \( t \) years.

\[
P' = rP \quad \text{with} \quad r > 0 \\
P' = rP \quad \text{with} \quad r < 0 \\
P' = rP(1 - P/K)
\]

c. Based on previous research, you know that the population of the cyanobacterium *Synechocystis* is approximately 12 hours. You are growing a colony of *Synechosystis* in a test tube and you want a function \( P(t) \) that models the population after \( t \) hours.

\[
P' = rP \quad \text{with} \quad r > 0 \\
P' = rP \quad \text{with} \quad r < 0 \\
P' = rP(1 - P/K)
\]

4. Consider the differential equation \( y' = (y + 1)(2y - 6)(y - 5) \).

a. What are the equilibria of this system?

The equilibria are the \( y \) values at which \( y' \) is equal to zero. Here, \( y = -1, y = 3, \) and \( y = 5 \) are the equilibria.

b. Which of the equilibria you found are stable? (It may help you to draw a diagram on the \( ty \)-plane provided on the 2nd page of the exam, but you do not have to for credit.) \( y' > 0 \) when \( -1 < y < 3 \) and when \( y > 5 \). On the other hand, \( y' < 0 \) when \( y < -1 \) and when \( 3 < y < 5 \). Drawing the direction field shows us that \( 3 \) is a stable equilibrium.

c. If \( y(t) \) is a solution to the diff. eq. above, and \( y(0) = 1 \), then what is \( \lim_{t \to \infty} y(t) \)?

Since \( y'(0) > 1 \), \( y \) will grow towards the stable equilibrium at \( y = 3 \). The limit is 3.

5. Let \( R \) be the region bounded above by the line \( y = 3x - 2 \) and the \( x \)-axis, and below by the parabola \( x^2 - 12 \).

Your friend Mip wants to know the area of this region. Mip has set up the integrals:

\[
A = \int_a^b [(3x - 2) - (x^2 - 12)] \, dx + \int_b^c [-x^2 - 12] \, dx
\]

but hasn’t figured out where the endpoints of integration should be. Help Mip out by finding the values for \( a, b, c \). *Draw a box around your final answer(s).*
DO NOT ATTEMPT TO COMPUTE $A$, JUST FIND THE ENDPOINTS.

The point $a$ is where the parabola and the slanted line intersect. To find this, we set them equal to each other and solve for $x$:

\[
x^2 - 12 = 3x - 2
\]
\[
x^2 - 3x - 10 = 0
\]
\[
(x - 5)(x + 2) = 0
\]

so either $x = -2$ or $x = 5$. Since the intersection point we care about is to the left of the $y$-axis, $a = -2$.

The point $b$ is where the slanted line and the $x$-axis intersect. This is where $3x - 2 = 0$, so $b = 2/3$.

The point $c$ is where the parabola and the $x$-axis intersect. There are two such points, $x = -\sqrt{12}$ and $x = \sqrt{12}$. Since $c$ is to the right of the $y$-axis, $c = \sqrt{12}$.

6. Let $R$ be the region bounded above by $y = 2\pi$, below by $y = \sqrt{1 - \cos x}$, on the left by $x = 0$, and on the right by $x = 2\pi$.

If \( \int_0^{2\pi} \sqrt{1 - \cos x} \, dx = 4\sqrt{2} \), then what is the area of $R$?

The area of $R$ is the area of the square minus the bump created by the curve $y = \sqrt{1 - \cos x}$.

Thus,

\[
A(R) = (2\pi)^2 - \int_0^{2\pi} \sqrt{1 - \cos x} \, dx
\]
\[
= 4\pi^2 - 4\sqrt{2}
\]

7. Let $R_1$ be the region bounded above by $y = \sqrt{1 + \sin x \cos x}$, below by the $x$-axis, on the left by $x = 0$, and on the right by $x = \pi$. Let $S_1$ be the solid obtained by revolving $R$ around the $x$-axis. Write down a definite integral which is equal to the volume of $S_1$.

Using the washers formula,

\[
V(S_1) = \int_0^\pi \pi(\sqrt{1 + \sin x \cos x})^2 \, dx = \int_0^\pi \pi(1 + \sin x \cos x) \, dx
\]
9. Find the volume of either $S_1$ (from Problem 7) or $S_2$ (from Problem 8). Your choice!

Circle which solid you’ve chosen: $S_1$  $S_2$, then find its area.

*Hint: Both integrals can be evaluated with $u$-substitution. Neither requires $IbP$.*

*Draw a box around your final answer.*

For $S_1$, 

$$
\int_0^\pi \pi (1 + \sin x \cos x) \, dx = \pi \int_0^\pi dx + \pi \int_0^\pi \sin x \cos x \, dx
$$

$$
= \pi^2
$$