Here are some tips:

- If you have time, it’s always a good idea to check your work.
- If you get the wrong answer for an integral but show your work, chances are good that we can award you partial credit.
- DO NOT attempt to estimate any of your answers as decimals. For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is exact.
- The boxes at the end of each topic are for grading purposes only. Do not touch or look at these boxes. Pretend they are not there.
- The last page of the exam is blank, and can be used for any extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

You will not need any integration table entries or trigonometric identities for this exam.

The figures below are for Problems 1 and 2, respectively:

Hint for Problem 2: An equilateral triangle with sidlength $s$ has area equal to $\frac{\sqrt{3}}{4} s^2$. 
Problem 1

Let $R_1$ be the region bounded above by the curves $y = \sqrt{x}$ and $y = 2\sqrt{1-x}$ and below by the $x$-axis. $R_1$ is illustrated on the second page of this exam.

Write down a definite integral or a sum of definite integrals equal to the area of $R_1$.

Do NOT evaluate the integral(s). Draw a box around your final answer.

Step 1: Find point of intersection.

\[
\sqrt{x} = 2\sqrt{1-x} \\
x = (2\sqrt{1-x})^2 = 4(1-x) \\
x = 4 - 4x \\
5x = 4 \\
x = 4/5
\]

Step 2: Write integral

\[
\text{Area} = \int_0^{4/5} \sqrt{x} \, dx + \int_{4/5}^1 2\sqrt{1-x} \, dx
\]

Problem 2

With $R_1$ as in Problem 1, let $S_1$ and $S'_1$ be solids with base $R_1$ such that

- The cross-sections of $S_1$ perpendicular to the $x$-axis are squares; and
- The cross-sections of $S'_1$ perpendicular to the $x$-axis are semicircles

How do the volumes of $S_1$ and $S'_1$ compare?

Note: You do NOT have to compute the volumes of these solids to find the correct answer.

For $S_1$, the volume will be the integral of side(x)^2 where side(x) is the distance between the upper and lower boundaries of $R_1$.

For $S'_1$, the volume will be the integral of $(\pi \times \text{radius}(x)^2)/2$ where radius(x) is half the distance between the upper and lower boundaries of $R_1$.

Since radius(x) = side(x)/2, in $S'_1$ we are integrating $\pi/8 \times (\text{side}(x)^2)$, so the volume of $S'_1$ will be $\pi/8$ times the volume of $S_1$.

The correct answer is (iv).
Problem 3

Let \( R_2 \) be the region bounded above by \( y = 2\sqrt{x} \), below by \( y = \sqrt{x} \), and on the right by \( x = 2 \). \( R_2 \) is illustrated on the second page of this exam.

Let \( S_2 \) be the solid obtained by revolving \( R_2 \) around the \( x \)-axis.

What is the volume of \( S_2 \)? (Yes, you DO need to evaluate the integral this time.)

**Draw a box around your final answer.**

\[
\text{Vol} (S_2) = \int_0^2 \pi \left[ (2\sqrt{x})^2 - (\sqrt{x})^2 \right] \, dx
\]

\[
= \int_0^2 \pi \left[ 4x - x \right] \, dx
\]

\[
= 3 \pi \int_0^2 x \, dx
\]

\[
= 3 \pi \left[ \frac{x^2}{2} \right]_0^2
\]

\[
= 3 \pi \left[ \frac{4}{2} - \frac{0}{2} \right]
\]

\[
= 6\pi
\]
Problem 4

a. A train traveling north at an initial velocity of 120 kilometers per hour begins to decelerate at time $t = 0$. Its deceleration is constant: $a(t) = -k$ (in kilometers per hour per hour) for some positive constant $k$. If the train stops moving at time $t = 1$ (one hour later), solve for $v(t)$, the train’s velocity function. Draw a box around your final answer.

\[
v(t) = \int a(t) \, dt = \int (-k) \, dt = -kt + C
\]

We know $v(0) = 120$ and $v(1) = 0$, so

\[
C = 120 \quad \text{and} \quad k = 120
\]

\[
v(t) = -120t + 120
\]

b. Let $y(t)$ be the position function of the train. Write down a definite integral whose value is equal to $y(1) - y(0)$, the distance traveled by the train over the course of the hour starting at $t = 0$. You do NOT need to evaluate this integral. Draw a box around your final answer.

\[
y(1) - y(0) = \int_0^1 y'(t) \, dt = \left( \int_0^1 v(t) \, dt \right) = \left( \int_0^1 (-120t + 120) \, dt \right)
\]
Problem 5

What is the name of the differential equation we studied that is used to model the growth of a population with a finite carrying capacity?

the logistic differential equation

Problem 6

Our friend Partario is attempting to solve the following initial value problem:

\[ \frac{dy}{dx} = 2\sqrt{y} \quad \text{with} \quad y(0) = 1. \]

His solution:

- **Step 1:** Separation of variables yields \( \frac{dy}{2\sqrt{y}} = dx. \)
- **Step 2:** Integrating both sides yields \( \sqrt{y} = x + C. \)
- **Step 3:** Solving for \( y \) yields \( y = x^2 + C. \)
- **Step 4:** Using the initial condition \( y(0) = 1 \) yields \( C = 1 \), so \( y = x^2 + 1. \)

Since he’s pretty good at math but short on confidence, Partario asks you to look over his work.

a. Is Partario correct? **Circle your answer.**

PARTARIO IS CORRECT    PARTARIO IS INCORRECT

b. What is the correct answer to Partario’s Problem? **You do not need to show your work or justify your answer.**

In **Step 3**, Partario should have gotten

\[ y = (x+C)^2 \]

In fact, \( y = (x+1)^2 \)

you can check that this works but \( y = x^2 + 1 \) does not by plugging into
Problem 7
The differential equation \( y' = y \) has a single equilibrium. Is this equilibrium stable, unstable, or metastable? Circle your answer.

\[ \begin{array}{ccc}
\text{STABLE} & & \text{UNSTABLE} \\
\text{METASTABLE} & & \\
\end{array} \]

Problem 8
Which of the following functions satisfy the second order differential equation \( y'' = 9y \)?

- a. \( y = \sin 3x \)
- b. \( y = \cos 3x \)
- c. \( y = 9e^x \)
- d. \( y = 2e^{-3x} \)
- e. \( y = e^{3x} \)
- f. None of them.

\[ y = e^{3x} \quad y' = 3e^{3x} \quad y'' = 9e^{3x} = 9y \]

Problem 9
Dead leaves accumulate on the ground in a forest at a rate of 6 grams per square centimeter per year. At the same time, these leaves decompose at a continuous rate of 75% per year.

a. Write down a differential equation for the total quantity of dead leaves \( L \) (per square centimeter) at time \( t \). Just write it down. Do not separate and solve for \( L(t) \).

\[ L' = 6 - 0.75L \]

b. Suppose \( L(t) \) is a solution to the differential equation from part (a) with \( L(0) \geq 0 \).

What is the value of \( \lim_{t \to \infty} L(t) \)?

Circle (i.) or (ii.) below, and fill in the blank if you choose (ii.).

- i. There is not enough information provided to determine the value of this limit.
- ii. The value of the limit is \( 8 \text{ (g/cm}^2) \) equilibrium: \( 6 - 0.75L = 0 \)

\[ 0.75L = 6 \]
\[ L = 8 \]