This homework assignment is meant to help us get back into the practice of taking integrals and limits, while exploring the ideas of Type I improper integrals.

**Problems 1-3:** Calculate the integral if it converges or show that it diverges.

**Problem 1:** $\int_{1}^{\infty} \frac{1}{(x+2)^2} \, dx$  \quad **Problem 2:** $\int_{1}^{\infty} \frac{x}{4 + x^2} \, dx$  \quad **Problem 3:** $\int_{1}^{\infty} \frac{y}{y^4 + 1} \, dy$

**Problem 4:** For this problem, $\alpha$ is a constant with $\alpha > 0$. Your answers may depend on $\alpha$. Calculate:

(a) $\int_{0}^{\infty} \frac{e^{-y/\alpha}}{\alpha} \, dy$  \quad (b) $\int_{0}^{\infty} \frac{ye^{-y/\alpha}}{\alpha} \, dy$  \quad (c) $\int_{0}^{\infty} \frac{y^2 e^{-y/\alpha}}{\alpha} \, dy$

**Problem 5:** The gamma function is defined for all $x > 0$ by the rule $\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt$.

(a) Find $\Gamma(1)$ and $\Gamma(2)$.

(b) Integrate by parts with respect to $t$ to show that for positive $n$, $\Gamma(n+1) = n\Gamma(n)$.

(c) Using parts (a) and (b), evaluate $\Gamma(3)$, $\Gamma(4)$, and $\Gamma(5)$ without evaluating more integrals. Do you see a pattern? Can you find a simple expression for $\Gamma(n)$ when $n$ is a positive integer?

**Problem 6:** Decide if the integral $\int_{1}^{\infty} \frac{2 + \cos(\phi)}{\phi^2} \, d\phi$ converges or diverges.

*Note: that symbol, $\phi$, is the Greek letter “phi.”*

**Problem 7:** Consider the integral $\int_{2}^{\infty} \frac{1}{x(\ln x)^p} \, dx$ where $p$ is a constant. For what values of $p$ does the integral converge? For what values of $p$ does the integral diverge?

**Problem 8:** Compute the improper integral $\int_{2}^{\infty} \frac{1}{(x+2)(x-1)} \, dx$ if it converges or show that it diverges.

*Hint: log rules will be helpful.*

[More problems on next page]
Problems 9 - 12: Compute the following limits if they exist or show that they do not exist. If a limit does not exist, explain the behavior of the function (e.g. diverges to $\pm \infty$ or oscillates or something else).

Problem 9: \[ \lim_{x \to \infty} \left( \frac{x^2}{x-1} - \frac{x^2}{x+2} \right) \]

Problem 10: \[ \lim_{t \to \pi} \frac{\sin^2 t}{t - \pi} \]

Problem 11: \[ \lim_{y \to 1} \cos(2y) \]

Problem 12: \[ \lim_{r \to 0} r \ln |r| \]

Here is a table of integrals, if you need it. Don’t remember table of integrals and/or never saw them? Section 7.3 of the book can help!

Table of potentially useful integrals

Note: This is the table that will be provided on every exam this quarter. Also remember that $a, b, c, d$ are meant to be constants – as in, they don’t depend on the variable of integration.

If you use one of these formulas, please cite it by number in your solution.

1. \[ \int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C, \, a \neq 0 \]

2. \[ \int \frac{bx + c}{x^2 + a^2} \, dx = \frac{b}{2} \ln |x^2 + a^2| + \frac{c}{a} \arctan \left( \frac{x}{a} \right) + C, \, a \neq 0 \]

3. \[ \int \frac{1}{(x-a)(x-b)} \, dx = \frac{1}{a-b} \left( \ln |x-a| - \ln |x-b| \right) + C, \, a \neq b \]

4. \[ \int \frac{cx + d}{(x-a)(x-b)} \, dx = \frac{1}{a-b} \left[ (ac+d) \ln |x-a| - (bc+d) \ln |x-b| \right] + C, \, a \neq b \]

5. \[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \left( \frac{x}{a} \right) + C, \, a > 0 \]

6. \[ \int \frac{1}{\sqrt{x^2 \pm a^2}} \, dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C, \, a > 0 \]

7. \[ \int \sqrt{a^2 \pm x^2} \, dx = \frac{1}{2} \left( x \sqrt{a^2 \pm x^2} + a^2 \int \frac{1}{\sqrt{a^2 \pm x^2}} \, dx \right) + C, \, a > 0 \]

8. \[ \int \sqrt{x^2 - a^2} \, dx = \frac{1}{2} \left( x \sqrt{x^2 - a^2} - a^2 \int \frac{1}{\sqrt{x^2 - a^2}} \, dx \right) + C, \, a > 0 \]