This homework assignment looks at some interesting things that happen with infinity and continues our exploration of series.

**Problems 1 and 2**: In these problems, we’ll consider an infinite solid of revolution. The shape is created by taking the region bounded by the curve \( y = \frac{1}{x} \) and the \( x \)-axis for \( x \geq 1 \) (so an infinite region) and rotating it around the \( x \)-axis. **You’ll need some geometry integral formulas that we have included on the last page of this document.**

**Problem 1**: First, let’s look at the region and curve used to create this object. Be sure to justify your answers.

*Note: We are not asking you to compute anything!*

(a) Write an improper integral that gives the area of the region under \( y = \frac{1}{x} \) and above the \( x \)-axis for \( x \geq 1 \). Does this integral converge or diverge?

(b) Write an improper integral that gives the arc length of the curve \( y = \frac{1}{x} \) for \( x \geq 1 \). Does this integral converge or diverge?

**Problem 2**: Now, let’s consider the actual solid of revolution. Be sure to justify your answers.

*Note: Again, we are not asking you to compute anything!*

(a) Write an improper integral that gives the surface area of the object. Does this integral converge or diverge?

(b) Write an improper integral that gives the volume of the object. Does this integral converge or diverge?

You don’t have to write anything about this but we recommend you take a few minutes to think about your answers in Problems 1 and 2.

**Problem 3**: Which of the following series are geometric? If the series is geometric, identify the common ratio. **You do not have to determine convergence/divergence of these series.**

(a) \[ \sum_{n=5}^{\infty} \frac{2n}{(1 + n^2)^2} \]  
(b) \[ \sum_{n=1}^{\infty} \frac{(-1)^n2^{3+n}}{3^n} \]  
(c) \[ \sum_{n=2}^{\infty} \frac{n}{3^n} \]  
(d) \[ \sum_{n=3}^{\infty} (\ln 2)^{3n} \]  
(e) \[ \sum_{n=1}^{\infty} \frac{5^n}{n^n} \]
Problem 4: One tool we had for making integration easier was substitution. For series, we have an analogous (though far more restrictive) tool called reindexing. Essentially, we can shift our index variable (the thing we are summing over) by integers and get a different looking formula that represents the same series. For example, we can see that

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)^3} = \sum_{m=3}^{\infty} \frac{1}{m^3} = \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots$$

by letting $m = n + 2$. When reindexing, we are only allowed to add or subtract an integer from our original index. This problem will explore this idea.

(a) Show that $\sum_{n=3}^{\infty} \frac{1}{(n-2)^2} = \sum_{m=1}^{\infty} \frac{1}{m^2}$.

(b) Show that $\sum_{n=1}^{\infty} \frac{2n + 1}{3^n} = \sum_{m=0}^{\infty} \frac{2m + 3}{3^{m+1}}$.

(c) Find a formula for $a_m$ so that $\sum_{n=1}^{\infty} \frac{1}{n^2 + 6n - 9} = \sum_{m=3}^{\infty} a_m$.

Problem 5: Use the integral test to determine if the series $\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$ converges or diverges. Be sure to justify that the integral test applies!

Problem 6: Find the sum of the series $\sum_{n=1}^{\infty} \left(2 \left(\frac{3}{5}\right)^n + 3 \left(\frac{4}{9}\right)^n\right)$ if it exists or show that the series diverges.

Problems 7-12: Determine whether the following series converge or diverge. For these problems, you may use what we know about geometric series, the integral test, and the comparison test. Clearly state which test you are using and be sure to justify that it applies.

Problem 7: $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{n^3 + 1}$

Problem 8: $\sum_{n=2}^{\infty} \frac{n + 2}{n^2 - 1}$

Problem 9: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{n-1}}{3^n}$

Problem 10: $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$

Problem 11: $\sum_{n=4}^{\infty} \frac{1}{n(1 + \ln n)}$

Problem 12: $\sum_{n=2}^{\infty} \frac{n}{n + 1}$
Geometry formulas for Problems 1 and 2

Arc length of $y = f(x)$ from $[a,b]$:

$$\int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

Surface area of a shape created by rotating a curve $y = f(x)$ from $x = a$ to $x = b$ about the $x$-axis:

$$\int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2} \, dx$$

Curious about where this formula comes from? It’s a great office hour question!

Volume of the solid obtained by rotating a region bounded by $y = f(x)$ and the $x$-axis from $x = a$ to $x = b$ about the $x$-axis:

$$\int_a^b \pi [f(x)]^2 \, dx$$

Table of potentially useful integrals

Note: This is the table that will be provided on every exam this quarter. Also remember that $a, b, c, d$ are meant to be constants – as in, they don’t depend on the variable of integration.

If you use one of these formulas, please cite it by number in your solution.

1. $\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C, a \neq 0$

2. $\int \frac{bx + c}{x^2 + a^2} \, dx = \frac{b}{2} \ln |x^2 + a^2| + \frac{c}{a} \arctan \left( \frac{x}{a} \right) + C, a \neq 0$

3. $\int \frac{1}{(x-a)(x-b)} \, dx = \frac{1}{a-b} (\ln |x-a| - \ln |x-b|) + C, a \neq b$

4. $\int \frac{cx + d}{(x-a)(x-b)} \, dx = \frac{1}{a-b} [(ac + d) \ln |x-a| - (bc + d) \ln |x-b|] + C, a \neq b$

5. $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \left( \frac{x}{a} \right) + C, a > 0$

6. $\int \frac{1}{\sqrt{x^2 \pm a^2}} \, dx = \ln |x + \sqrt{x^2 \pm a^2}| + C, a > 0$

7. $\int \sqrt{a^2 \pm x^2} \, dx = \frac{1}{2} \left( x\sqrt{a^2 \pm x^2} + a^2 \int \frac{1}{\sqrt{a^2 \pm x^2}} \, dx \right) + C, a > 0$

8. $\int \sqrt{x^2 - a^2} \, dx = \frac{1}{2} \left( x\sqrt{x^2 - a^2} - a^2 \int \frac{1}{\sqrt{x^2 - a^2}} \, dx \right) + C, a > 0$