Math 19 review – solutions

HOW TO USE THIS: try the problems on your own. When you’re done (or you get stuck), check things over here. DO NOT just read the solution. Copy it down; write it in your own words; put it away and try the problem again. Just reading the solution will not help you learn it. You have to try it out again on your own to make sure it sticks.

(1) Find the derivative of the following functions.

(a) $e^x(x + x\sqrt{x})$

**Solution:** we use product rule and the fact that $x\sqrt{x} = x^{3/2}$ to get

$$\frac{d}{dx} e^x(x + x\sqrt{x}) = e^x(x + x\sqrt{x}) + e^x \left(1 + \frac{3}{2} \sqrt{x}\right) = e^x \left(1 + \frac{3\sqrt{x}}{2} + x + x\sqrt{x}\right).$$

(b) $x^{7/6} - \ln(x^2 + 1)$

**Solution:** we use chain rule and power rule to get

$$\frac{d}{dx} x^{7/6} - \ln(x^2 + 1) = \frac{7}{6} x^{1/6} - \frac{1}{x^2 + 1} (2x) = \frac{7x^{1/6}}{6} - \frac{2x}{x^2 + 1}.$$

(c) $3^{-x^2}$

**Solution:** two ways to do this one. First, we could recall that $\frac{d}{dx} a^x = a^x (\ln a)$ and use that, with the chain rule, since the base 3 is a constant:

$$\frac{d}{dx} 3^{-x^2} = 3^{-x^2} (\ln 3)(-2x).$$

Alternatively, we could rewrite $3 = e^{\ln 3}$ and just use chain rule

$$\frac{d}{dx} 3^{-x^2} = \frac{d}{dx} e^{-x^2 (\ln 3)} = e^{-x^2 (\ln 3)} (-2(\ln 3)x) = 3^{-x^2} (-2(\ln 3)x).$$

We get the same answer using both methods.

(d) $(x^3 - 1) \sin(x)$

**Solution:** we use the product rule to see that

$$\frac{d}{dx} (x^3 - 1) \sin(x) = 3x^2 \sin(x) + (x^3 - 1) \cos(x).$$

(e) $2^{40}$

**Solution:** $2^{40}$ is a constant, so its derivative is 0.

(f) $\frac{e^{1/x}}{x^2}$

**Solution:** we could use quotient rule or we could rewrite the function slightly and use product rule

$$\frac{d}{dx} \frac{e^{1/x}}{x^2} = \frac{d}{dx} e^{1/x} x^{-2} = e^{1/x} \left(-\frac{1}{x^2}\right) x^{-2} + e^{1/x} (-2x^{-3}) = e^{1/x} \left(-\frac{1}{x^4} - \frac{2}{x^3}\right).$$
(g) $\arctan\left(\frac{x^2 + 3}{x - 1}\right)$

**Solution:** Here we use the chain rule and the quotient rule

$$\frac{d}{dx} \arctan\left(\frac{x^2 + 3}{x - 1}\right) = \frac{1}{1 + \left(\frac{x^2 + 3}{x - 1}\right)^2} \left(\frac{d}{dx} \frac{x^2 + 3}{x - 1}\right)$$

$$= \frac{(x - 1)^2}{(x - 1)^2 + (x^2 + 3)^2} \left(\frac{2x(x - 1) - (x^2 + 3)}{(x - 1)^2}\right)$$

$$= \frac{(x - 1)^2(x^2 - 2x - 3)}{(x - 1)^2 + (x^2 + 3)^2(x - 1)^2}$$

$$= \frac{(x - 3)(x + 1)}{(x - 1)^2 + (x^2 + 3)^2}$$

Note: you don’t have to simplify but it does turn out rather nice after you do.

(2) Compute the following limits using any technique you think appropriate. Make sure to JUSTIFY your answer by showing your work.

(a) $\lim_{x \to 0} \frac{\sin(4x)}{\tan(5x)}$

**Solution:** This becomes $\frac{0}{0}$, so we can use L’Hopital:

$$\lim_{x \to 0} \frac{\sin(4x)}{\tan(5x)} = \lim_{x \to 0} \frac{4 \cos(4x)}{5 \sec^2(5x)} = \lim_{x \to 0} \frac{4}{5} \cos(4x) \cos^2(5x) = \frac{4}{5}$$

(b) $\lim_{x \to 3} \left(1 + \frac{1}{x}\right)^2$

**Solution:** Nothing fancy here – just plug in $x = 3$ and you’re good! The limit is $\frac{16}{9}$.

(c) $\lim_{x \to -\infty} \arctan\left(\frac{x^2 - 1}{x^2 + 1}\right)$

**Solution:** The key fact is that if $\lim_{x \to a} g(x) = L$, and $f$ is continuous at $L$, then $\lim_{x \to a} f(g(x)) = f(L)$. Or more generally (allowing $L$ to be $\infty$ or $-\infty$), if $\lim_{x \to a} g(x) = L$, and $\lim_{x \to L} f(x)$ exists, then $\lim_{x \to a} f(g(x)) = \lim_{x \to L} f(x)$.

The limit $\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$. Since $\arctan(1) = \frac{\pi}{4}$, $\lim_{x \to -\infty} \arctan\left(\frac{x^2 - 1}{x^2 + 1}\right) = \frac{\pi}{4}$.

(d) $\lim_{x \to \infty} \frac{\ln(x^2 + 5x + 1)}{\ln(x^3 - 10x)}$

**Solution:** We give two different solutions. Read both, then focus on the one you feel more comfortable with.
The first solution is to factor out the highest power of $x$ inside the logarithms:

$$\frac{\ln(x^2 + 5x + 1)}{\ln(x^3 - 10x)} = \frac{\ln(x^2(1 + 5/x + 1/x^2))}{\ln(x^3(1 - 10/x^2))}$$

$$= \frac{2\ln(x) + \ln(1 + 5/x + 1/x^2)}{3\ln(x) + \ln(1 - 10/x^2)}$$

$$= \frac{2 + \frac{\ln(1 + 5/x + 1/x^2)}{\ln(x)}}{3 + \frac{\ln(1 - 10/x^2)}{\ln(x)}}$$

Thus the limit is $2/3$. Another way to solve this problem is to use L’hopital’s rule, which applies since we have an $\infty/\infty$ form. Doing this we obtain that the limit in question is equal to the limit

$$\lim_{x \to \infty} \frac{(2x + 5)/(x^2 + 5x + 1)}{(3x^2 - 10)/(x^3 - 10x)} = \lim_{x \to \infty} \frac{(2x + 5)(x^3 - 10x)}{(x^2 + 5x + 1)(3x^2 - 10)}$$

$$= \lim_{x \to \infty} \frac{2x^4 + \ldots}{3x^4 + \ldots}$$

$$= \frac{2}{3}.$$

(e) $\lim_{x \to -\infty} \left( \sqrt{x^2 + 4x + 1} - x \right)$

Solution: As $x \to -\infty$, $\sqrt{x^2 + 4x + 1} \to \infty$ and $-x \to \infty$, meaning the function starts to look like a huge number plus another huge number. This means the limit goes to $\infty$.

(f) $\lim_{x \to -\infty} \ln \left( 1 + \frac{\sin(x)}{x} \right)$

Solution: Since $\ln(x)$ is a continuous function, we can start by computing $\lim_{x \to -\infty} 1 + \frac{\sin(x)}{x}$. If we try to plug in, we run into problems because $\lim_{x \to -\infty} \sin(x)$ DNE. Instead, we use the Squeeze Theorem:

$$-1 \leq \sin x \leq 1$$

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

$$1 - \frac{1}{x} \leq 1 + \frac{\sin x}{x} \leq 1 + \frac{1}{x}$$

As $x \to -\infty$, $1 - \frac{1}{x}$ goes to 1 and $1 + \frac{1}{x}$ also goes to 1. Therefore, by the squeeze theorem,

$$\lim_{x \to -\infty} 1 + \frac{\sin x}{x} = 1.$$

Then we have to remember that this was inside of that natural log, so we finish off the problem by computing:

$$\lim_{x \to -\infty} \ln \left( 1 + \frac{\sin x}{x} \right) = \ln \left( \lim_{x \to -\infty} \left( 1 + \frac{\sin x}{x} \right) \right) = \ln(1) = 0$$
(g) \( \lim_{x \to \frac{\pi}{2}} \frac{\cos(x) + \cos(2x)}{\sin^2(x)} \)

**Solution:** Another simple one. If we plug in \( \frac{\pi}{2} \), we get

\[
\frac{\cos \left( \frac{\pi}{2} \right) + \cos(\pi)}{\sin^2 \left( \frac{\pi}{2} \right)} = \frac{0 - 1}{1} = -1
\]

(h) \( \lim_{x \to \infty} e^{k(x)} \) where \( k(x) = \frac{-x^3 + 4x}{x^2 + 10x + 1} \).

**Solution:** Since exponentiation is continuous, we can start by finding

\[
\lim_{x \to \infty} k(x) = \lim_{x \to \infty} \frac{-x^3 + 4x}{x^2 + 10x + 1}
\]

\[
= \lim_{x \to \infty} \frac{x^3 \left( -1 + \frac{4}{x} \right)}{x^2 \left( 1 + \frac{10}{x} + \frac{1}{x^2} \right)}
\]

\[
= \lim_{x \to \infty} \frac{x \left( -1 + \frac{4}{x} \right)}{\left( 1 + \frac{10}{x} + \frac{1}{x^2} \right)}
\]

At this point, we see that the numerator starts to look like \(-\infty\) as \( x \to \infty \) while the denominator starts to look like 1. This means

\[
\lim_{x \to \infty} k(x) = -\infty.
\]

But our question was to find

\[
\lim_{x \to \infty} e^{k(x)} = e^{\lim_{x \to \infty} k(x)} = e^{-\infty} = 0.
\]