Math 20 review – solutions

HOW TO USE THIS: try the problems on your own. When you’re done (or you get stuck), check things over here. **DO NOT** just read the solution. Copy it down; write it in your own words; put it away and try the problem again. Just reading the solution will not help you learn it. You have to try it out again on your own to make sure it sticks.

(1) \[ \int_1^4 \frac{\sqrt{y} - y}{y^2} \, dy \]

**Solution:** Let’s start off slowly. What we see here: a fraction with a denominator and a subtraction sign in the numerator. So we can split the fraction, cancel our powers of \( y \) and go from there:

\[
\begin{align*}
\int_1^4 \frac{\sqrt{y} - y}{y^2} \, dy &= \int_1^4 \frac{\sqrt{y}}{y^2} - \frac{y}{y^2} \, dy \\
&= \int_1^4 \frac{1}{y^{3/2}} - \frac{1}{y} \, dy \\
&= \left. -\frac{2}{y^{1/2}} - \ln |y| \right|_1^4 \\
&= \frac{-2}{2} - \ln(4) - (-2 - \ln(1)) = 1 - \ln(4)
\end{align*}
\]

(2) \[ \int x^3 \sqrt{x^2 + 1} \, dx \]

**Solution:** We have complicated thing \((x^2 + 1)\) under a square root, which means our first thought should be to simplify that with a substitution. Letting \(u = x^2 + 1\), we have that \(du = 2x \, dx\), meaning \(x^3 \, dx = (x^2) x \, dx = \frac{1}{2} x^2 \, du\). How do we deal with the leftover \(x^2\)? Well, since \(u = x^2 + 1\), \(x^2 = u - 1\). Using this, we can complete the substitution and evaluate the integral.

\[
\begin{align*}
\int x^3 \sqrt{x^2 + 1} \, dx &= \frac{1}{2} \int (u - 1) \sqrt{u} \, du \\
&= \frac{1}{2} \int u^{3/2} - \sqrt{u} \, du \\
&= \frac{1}{2} \left( \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \\
&= \frac{1}{5} (x^2 + 1)^{5/2} - \frac{1}{3} (x^2 + 1)^{3/2} + C
\end{align*}
\]
\( \int \frac{w}{w - 2} \, dw \)

**Solution:** Using the substitution \( u = w - 2 \) \((w = u + 2)\), \( du = dw \), we get

\[
\int \frac{w}{w - 2} \, dw = \int \frac{u + 2}{u} \, du = \int \left[ 1 + \frac{2}{u} \right] \, du = u + \ln |u| + C = (w - 2) + \ln |w - 2| + C
\]

**NOTE:** Since \( C \) is any constant, \( w + \ln |w - 2| + C \) is also a form of the most general antiderivative.

\( \int r^4 \ln(r) \, dr \)

**Solution:** You can try starting this one with a substitution – \( u = \ln r \) so \( du = \frac{1}{r} \, dr \) and \( r = e^u \) – but you might be too happy about that idea. And you’ll end up needing to do integration by parts afterwards anyway.

Instead, let’s just dive right in with integration by parts. Remember we want to choose \( f(r) \) and \( g'(r) \) so that \( g'(r) \) is something we can actually integrate and so that \( f(r)g'(r) = r^4 \ln(r) \).

Here, let’s take

\[
f(r) = \ln r \quad g(r) = \frac{r^5}{5} \\
f'(r) = \frac{1}{r} \quad g'(r) = r^4
\]

This gives us

\[
\int r^4 \ln(r) \, dr = \frac{r^5 \ln(r)}{5} - \frac{1}{5} \int \frac{r^5}{r} \, dr = \frac{r^5 \ln(r)}{5} - \frac{r^5}{25} + C
\]

where we canceled the power of \( r \) before integrating the second integral.

\( \int \cos(\sqrt{t}) \, dt \)

**Solution:** Looking at this integral, we see a composition of two functions. And what’s more, that composition is making what would be a simple integral, \( \cos(u) \), more complicated. Enter substitution! Let’s see if substituting \( u = \sqrt{t} \) will help us:

\[
u = \sqrt{t} \quad \Rightarrow \quad du = \frac{1}{2\sqrt{t}} \, dt \\
2\sqrt{t} \, du = dt \\
2u \, du = dt
\]
Carrying out the substitution gives

\[ \int \cos(\sqrt{t}) \, dt = 2 \int u \cos(u) \, du \]

Now we have the product of two functions, which means integration by parts!

\[
\begin{align*}
  f(u) &= u & g(u) &= \sin(u) \\
  f'(u) &= 1 & g'(u) &= \cos(u)
\end{align*}
\]

Putting it all together, we can now compute the integral:

\[
\int \cos(\sqrt{t}) \, dt = 2 \int u \cos(u) \, du = 2 \left( u \sin(u) - \int \sin(u) \, du \right) = 2u \sin(u) + 2 \cos(u) + C = 2\sqrt{t} \sin(\sqrt{t}) + 2 \cos(\sqrt{t}) + C
\]

(6) \[ \int \frac{x^2}{\sqrt{1 + x^3}} \, dx \]

**Solution:** We can simplify the denominator with the substitution \( u = x^3 + 1 \), \( du = 3x^2 \, dx \), giving

\[
\int \frac{x^2}{\sqrt{1 + x^3}} \, dx = \frac{1}{3} \int \frac{1}{\sqrt{u}} \, du = \frac{2}{3} \sqrt{u} + C = \frac{2}{3} \sqrt{1 + x^3} + C
\]

(7) \[ \int xe^{-x^2} \, dx \]

**Solution:** This is a somewhat straightforward substitution as well: \( u = -x^2 \), \( du = -2x \, dx \):

\[
\int xe^{-x^2} \, dx = -\frac{1}{2} \int e^u \, du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-x^2} + C
\]

(8) \[ \int \frac{\ln x}{x} \, dx \]

**Solution:** Using the substitution \( u = \ln x \), \( du = \frac{1}{x} \, dx \):

\[
\int \frac{\ln x}{x} \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{1}{2} (\ln x)^2 + C
\]
(9) \[ \int \frac{t}{t^4 + 2} \, dt \]

**Solution:** Here, a little bit of rewriting goes a long way. In particular, \( t^4 = (t^2)^2 \). So our denominator is really something squared plus a constant and we can really see that by substituting \( u = t^2, \ du = 2t \, dt \):

\[
\int \frac{t}{t^4 + 1} \, dt = \frac{1}{2} \int \frac{1}{u^2 + 2} \, du.
\]

Now we can use the table to get:

\[
\frac{\sqrt{2}}{4} \arctan \left( \frac{t^2}{\sqrt{2}} \right) + C.
\]

(10) \[ \int (\tan x) \ln(\cos x) \, dx \]

**Solution:** First thing you might notice is the \( \cos(x) \) inside a logarithm and think, “composition? Maybe try substitution.” So \( u = \cos(x), \ du = -\sin(x) \, dx \). When you go to put that in, suddenly you see, “Wait. We don’t have \( \sin(x) \). We have \( \tan(x) \)...” Just remember that \( \tan(x) = \frac{\sin(x)}{\cos(x)} \), so the \( \sin(x) \) that we want to see is actually there. Thus, the substitution gives us

\[
\int (\tan x) \ln(\cos x) \, dx = \int \frac{\sin(x)}{\cos(x)} \ln(\cos x) \, dx = -\int \frac{\ln u}{u} \, du
\]

To deal with the new integral we have, we do another substitution: \( v = \ln u, \ dv = \frac{1}{u} \, du \) and get

\[
-\int \frac{\ln u}{u} \, du = -\int v \, dv = -\frac{v^2}{2} + C
\]

Putting things back in terms of the original variable of integration gives

\[
\int (\tan x) \ln(\cos x) \, dx = -\frac{(\ln(\cos x))^2}{2} + C
\]

(11) \[ \int (x^2 + 2x) \cos x \, dx \]

**Solution:** Product of two functions? Integration by parts it is!

\[
f(x) = x^2 + 2x \quad g(x) = \sin(x) \\
f'(x) = (2x + 2) \quad g'(x) = \cos(x)
\]

This gives us

\[
\int (x^2 + 2x) \cos x \, dx = (x^2 + 2x) \sin(x) - \int (2x + 2) \sin(x) \, dx
\]
Now: integration by parts again!

\[ f(x) = 2x + 2 \quad g(x) = -\cos(x) \]
\[ f'(x) = 2 \quad g'(x) = \sin(x) \]

This gives us

\[
\int (x^2 + 2x) \cos x \, dx = (x^2 + 2x) \sin(x) - \int (2x + 2) \sin(x) \, dx
\]
\[
= (x^2 + 2x) \sin(x) - \left(- (2x + 2) \cos(x) - 2 \int (- \cos(x)) \, dx \right)
\]
\[
= (x^2 + 2x) \sin(x) + (2x + 2) \cos(x) - 2 \int \cos x \, dx
\]
\[
= (x^2 + 2x) \sin(x) + (2x + 2) \cos(x) - 2 \sin(x) + C
\]

NOTE: you could have started by splitting the integral as

\[
\int (x^2 + 2x) \cos x \, dx = \int x^2 \cos x \, dx + 2 \int x \cos x \, dx
\]

and then done integrations by parts for each integral (twice for the first one, once for the second one).

(12) \[ \int \frac{x + 2}{\sqrt{x^2 + 4x}} \, dx \]

**Solution:** Let’s start with trying to simplify our denominator by making the substitution \( u = x^2 + 4x \). Notice that then \( du = (2x + 4) \, dx \), which is exactly twice the numerator. Therefore:

\[
\int \frac{x + 2}{\sqrt{x^2 + 4x}} \, dx = \frac{1}{2} \int \frac{1}{\sqrt{u}} \, du
\]
\[
= \sqrt{u} + C = \sqrt{x^2 + 4x} + C
\]

(13) \[ \int_0^1 x(\sqrt[3]{x} - \sqrt[4]{x}) \, dx \]

**Solution:** First, remember that \( \sqrt[n]{x} = x^{1/n} \). Then we distribute the \( x \) and integrate:

\[
\int_0^1 x(\sqrt[3]{x} - \sqrt[4]{x}) \, dx = \int_0^1 x^{4/3} - x^{5/4} \, dx
\]
\[
= \left[ \frac{3}{7} x^{7/3} - \frac{4}{9} x^{9/4} \right]_0^1
\]
\[
= \frac{3}{7} - \frac{4}{9} = -\frac{1}{63}
\]
(14) \( \int_0^2 (2x - 3)(4x^2 + 1) \, dx \)

**Solution:** You don’t have to do anything too complicated except for FOIL. (Well, you could do integration by parts if you really want to.)

\[
\int_0^2 (2x - 3)(4x^2 + 1) \, dx = \int_0^2 8x^3 - 12x^2 + 2x - 3 \, dx \\
= 2x^4 - 4x^3 + x^2 - 3x_0^2 \\
= 2(16) - 4(8) + 4 - 6 = -2
\]

(15) \( \int \frac{1}{x^2 - 6x + 5} \, dx \)

**Solution:** Back, again, to more involved integrals. We have a quadratic that we can factor in the denominator: \( x^2 - 6x + 5 = (x - 1)(x - 5) \). Using the table with \( a = 1 \) and \( b = 5 \) we have

\[
\int \frac{1}{(x - 1)(x - 5)} \, dx = -\frac{1}{4} \ln |x - 1| + \frac{1}{4} \ln |x - 5| + C
\]

(16) \( \int \frac{\ln x}{\sqrt{x}} \, dx \)

**Solution:** We’ll use integration by parts to find this antiderivative:

\[
f(x) = \ln x \quad g(x) = 2\sqrt{x} \\
f'(x) = \frac{1}{x} \quad g'(x) = \frac{1}{\sqrt{x}}
\]

Using this, we get

\[
\int \frac{\ln x}{\sqrt{x}} \, dx = 2\sqrt{x} \ln x - 2 \int \frac{1}{\sqrt{x}} \, dx \\
= 2\sqrt{x} \ln x - 2 \frac{1}{\sqrt{x}} \, dx \\
= 2\sqrt{x} \ln x - 4\sqrt{x} + C
\]

(17) \( \int \frac{1}{v^2 + 2v - 3} \, dv \)

**Solution:** Last one!* Just like in (15), we see a quadratic in the denominator. Makes sense to start by trying to factor† the denominator, which we can do:

\[v^2 + 2v - 3 = (v + 3)(v - 1)\]

Using the table with \( a = -3 \) and \( b = 1 \) we have

\[
\int \frac{1}{(v + 3)(v - 1)} \, dx = -\frac{1}{4} \ln |v + 3| + \frac{1}{4} \ln |v - 1| + C
\]

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*Or not. Maybe you didn’t go in order. Which is great! I just went in order while writing these solutions so...
†If we can’t factor it, our next option would be to complete the square.