Improper integrals

An improper integral is a definite integral \( \int_a^b f(x) \, dx \) where

I. One (or both) of \( a \) and \( b \) are \( \pm \infty \); and/or
II. the function \( f(x) \) is undefined for some value \( x \) in \( [a, b] \) (that is, with \( a \leq x \leq b \)).

Here are some examples of improper integrals:

\[
\begin{align*}
\text{a)} & \quad \int_1^\infty \frac{1}{x} \, dx \\
\text{b)} & \quad \int_0^1 \frac{1}{x} \, dx \\
\text{c)} & \quad \int_{-\infty}^\infty e^{-x^2} \, dx \\
\text{d)} & \quad \int_0^\infty \frac{1}{3x-1} \, dx
\end{align*}
\]

Explain why each of these integrals is improper, listing all the issues of types (I.) and/or (II.) that each integral has.

a) Type I only \( b/c \) upper limit is \( \infty \) and function defined on \([1, \infty)\)
b) Type II only; undefined at \( x=0 \), \( 0 \in [0, 1] \)
c) Type I only \( b/c \) \( \frac{\infty}{0} \), \( b=\infty \) and function defined on \(( -\infty, \infty )\)
d) Type I & Type II \( b/c \) \( b=\infty \) & function undef. at \( x=\frac{1}{3}, \frac{1}{3} \in [0, \infty) \)

Evaluating improper integrals with \( \infty \) as the upper bound

If \( f(x) \) is "nice" (defined and continuous, for example—no asymptotes!) on the interval \([a, \infty)\), then to evaluate \( \int_a^\infty f(x) \, dx \) we can follow a simple three-step process:

1. Rewrite the improper integral as a limit of proper integrals:

\[
\int_a^\infty f(x) \, dx = \lim_{b \to \infty} \left[ \int_a^b f(x) \, dx \right]
\]

2. Evaluate the proper integrals inside the limit.

3. Evaluate the limit as \( b \to \infty \).
Note that you may prefer to evaluate the proper integral first (reversing steps 1 and 2)—Dr. Solis uses this approach in his notes from Lecture 2. You are always welcome to use the method that you find easiest or most natural/comfortable.

**Example.** Evaluate \( \int_5^\infty \frac{1}{x^2} \, dx \).

1. Rewrite as limit: \( \int_5^\infty \frac{1}{x^2} \, dx = \lim_{b \to \infty} \int_5^b \frac{1}{x^2} \, dx \).

2. Evaluate the proper integrals (and simplify):

\[
\lim_{b \to \infty} \int_5^b \frac{1}{x^2} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{x} \right]_5^b = \lim_{b \to \infty} \left[ \frac{1}{b} - \frac{1}{5} \right] = \lim_{b \to \infty} \left[ \frac{1}{5} - \frac{1}{b} \right]
\]

3. Evaluate the limit: \( \lim_{b \to \infty} \left[ \frac{1}{5} - \frac{1}{b} \right] = \frac{1}{5} - 0 = \frac{1}{5} \).

**Convergence versus divergence**

When an improper integral is equal to a (finite) numerical value, we say that it converges. Otherwise, we say that it diverges.

**Examples:**

- \( \int_1^\infty \frac{1}{x} \, dx = \lim_{b \to \infty} \ln(b) = +\infty \) so this integral diverges (to positive infinity).

- \( \int_5^\infty \frac{1}{x^2} \, dx = 1/5 \) so this integral converges (to 1/5).

- \( \int_0^\infty \cos x \, dx = \lim_{b \to \infty} [\sin x] = \text{DNE} \) so this integral diverges.

- What about \( \int_0^\infty e^{-x} \, dx? \int_1^\infty \frac{1}{x^2} \, dx? \)

\[
\int_0^\infty e^{-x} \, dx = \lim_{b \to \infty} \int_0^b e^{-x} \, dx = \lim_{b \to \infty} [-e^{-x}]_0^b = \lim_{b \to \infty} (1 - e^{-b}) = 1
\]

This integral converges.

\[
\int_1^\infty \frac{1}{x^2} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{2x^2} \right]_0^b = \lim_{b \to \infty} \left( \frac{1}{2} - \frac{1}{2b^2} \right) = \frac{1}{2}
\]

This integral converges.
The asymptotic relation $\asymp$

If $f(x)$ and $g(x)$ are reasonably “nice” functions on $[a, \infty)$, then we say that $f(x)$ and $g(x)$ are asymptotic to each other (as $x \to \infty$) if the limit

$$\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right|$$

exists and is equal to a positive, finite constant $c$ (that is, $0 < c < \infty$). In symbols, we write $f(x) \asymp g(x)$.

Intuitively, $f(x) \asymp g(x)$ means that there is some constant $c$ (the same $c$ that pops out of the limit) such that $f(x) \approx c \cdot g(x)$ for large values of $x$.

Examples:

- $x^2 \asymp 4x^2 + 1$, since $\lim_{x \to \infty} \left| \frac{x^2}{4x^2 + 1} \right| = \frac{1}{4}$.
  
  *The intuition is that $x^2 \approx \frac{1}{4}(4x^2 + 1)$ for large values of $x$. Do you agree? What are the values of the left and right hand sides of the $\approx$ when $x = 100$?

  \[100^2 = 10,000\]
  \[\frac{1}{4}(4 \cdot 100^2 + 1) = 10,000.25\] pretty close!

- $\frac{1}{x} \asymp \frac{x}{3x^2 - 1}$ since $\lim_{x \to \infty} \left| \frac{1/x}{x/(3x^2 - 1)} \right| = \lim_{x \to \infty} \left| \frac{3x^2 - 1}{x^2} \right| = 3$

- $e^{-x} \not\asymp 3^{-x}$, because $\lim_{x \to \infty} \left| \frac{e^{-x}}{3^{-x}} \right| = \lim_{x \to \infty} \left( \frac{3}{e} \right)^x = \infty$.

Application in Math 21

The reason we care about the $\asymp$ relation in Math 21 is that it can sometimes allows us to quickly determine whether an improper integral converges or diverges, without actually evaluating it. Instead, we can compare a complicated integrand to a simpler one using $\asymp$, and then determine whether the simpler integral converges or diverges!

*Limit comparison for integrals.* If $f(x)$ and $g(x)$ are defined and continuous on the interval $[a, \infty)$ and $f(x) \asymp g(x)$ (as $x \to \infty$), then the improper integrals

$$\int_a^\infty f(x) \, dx \quad \text{and} \quad \int_a^\infty g(x) \, dx$$

either both converge or both diverge.
Limit comparison practice

Below, use limit comparison for integrals to determine whether the improper integral converges or diverges. You should state what ≈ relation you are using to establish this!

a. Does \( \int_1^\infty \frac{1}{3x-1} \, dx \) converge or diverge?

\[
guess \frac{1}{3x-1} \lessapprox \frac{1}{x} \quad \text{check: } \lim_{x \to \infty} \left| \frac{\frac{1}{3x-1}}{\frac{1}{x}} \right| = \lim_{x \to \infty} \left| \frac{x}{3x-1} \right| = \lim_{x \to \infty} \left| \frac{\frac{1}{3x-1}}{x} \right| = \frac{1}{3} \]

\( \frac{1}{3} \) is a positive constant so \( \frac{1}{3x-1} \lessapprox \frac{1}{x} \)

\( \int_1^\infty \frac{1}{x} \, dx \) Diverges (from examples page 2) so \( \int_1^\infty \frac{1}{3x-1} \, dx \) Diverges

b. Does \( \int_1^\infty \frac{1}{4x^2 + 1} \, dx \) converge or diverge?

\[
guess \frac{1}{4x^2 + 1} \lessapprox \frac{1}{4x^2} \quad \text{check: } \lim_{x \to \infty} \left| \frac{\frac{1}{4x^2 + 1}}{\frac{1}{4x^2}} \right| = \lim_{x \to \infty} \left| \frac{4x^2}{4x^2 + 1} \right| = \lim_{x \to \infty} \left| \frac{\frac{1}{4x^2 + 1}}{\frac{1}{4x^2}} \right| = 1 \]

\( \frac{1}{4x^2 + 1} \lessapprox \frac{1}{4x^2} \)

\( \int_1^\infty \frac{1}{4x^2} \, dx = \frac{1}{4} \int_1^\infty \frac{1}{x^2} \, dx \) converges (from examples page 2)

so \( \int_1^\infty \frac{1}{4x^2 + 1} \, dx \) converges

c. Does \( \int_0^\infty \frac{1}{e^x \sqrt{x^2 + x}} \, dx \) converge or diverge?

\[
guess \frac{\sqrt{x^2 + x}}{e^x} \lessapprox \frac{1}{e^x} \quad \text{check: } \lim_{x \to \infty} \left| \frac{\frac{\sqrt{x^2 + x}}{e^x}}{\frac{1}{e^x}} \right| = \lim_{x \to \infty} \left| \frac{\sqrt{x^2 + x}}{x^2 + x} \right| = \lim_{x \to \infty} \left| \frac{\frac{\sqrt{x^2 + x}}{x^2 + x}}{\frac{1}{e^x}} \right| = \frac{1}{e} \]

so since \( \int_0^\infty \frac{1}{e^x} \, dx \) converges (work on page 2)

we conclude \( \int_0^\infty \frac{\sqrt{x^2 + x}}{e^x} \, dx \) converges

so \( \int_0^\infty \frac{1}{4x^2 + 1} \, dx \) converges or diverge? (Compare with c.)

Integral converges. See explanation on Piazza.