Math 21

Lecture 4

Some review:

\[
\text{infinite sums } \quad \text{& improper } \int s \rightarrow \quad \text{converge: } \quad \text{diverge: } \\
\text{there is a finite answer } \quad \text{infinity, or } \quad \text{otherwise doesn't converge}
\]

The basic tool to address convergence/divergence issues is the asymptotic relation \( \asymp \). Note that \( \asymp \) is just one example of an asymptotic relation. Today we will see 2 more.

some more review: Suppose \( f, g \) are continuous and defined on \([a, \infty)\) then \( f \asymp g \) if

\[
\lim_{x \to \infty} \left| \frac{f}{g} \right| = c > 0
\]

Examples:

\[
3x^5 + x + 1 \asymp x^5
\]

\[
\frac{x^3}{x^2 + x + 1} \asymp x
\]

\[
\frac{\cos^2 x}{x^4 - x + 1} \asymp \frac{1}{x^4}
\]

Is the following statement true?

\[e^{-x^2} + x + 1 \asymp e^{-x^2}\]

The answer is NO! We set up the appropriate limit:

\[
\lim_{x \to \infty} \left| \frac{e^{-x^2} + x + 1}{e^{-x^2}} \right| = \lim_{x \to \infty} \left| e^{-x^2} + x + 1 \cdot e^{x^2} \right| = \lim_{x \to \infty} |e^{x+1}| = \infty
\]

Comparing the growth/decay of functions is a surprisingly useful tool. It is used to evaluate the efficiency of an algorithm in computer science; in this context you will hear it called big O notation. In general understanding growth and decay of functions is essential to quickly answering questions about long term behavior.

Here is a picture of functions we will work with
<table>
<thead>
<tr>
<th>Growth Rate</th>
<th>Example</th>
<th>value at $\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>exponential growth</td>
<td>$e^x, 2x^2 - 1, x^3$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>polynomial growth</td>
<td>$x^3 + 7x$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>slow growth</td>
<td>$\ln(x)$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>essentially constant</td>
<td>$\frac{x^3 - x}{x^3 + 1}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>slow decay</td>
<td>$\frac{1}{\ln(x)}$</td>
<td>0</td>
</tr>
<tr>
<td>polynomial decay</td>
<td>$\frac{1}{x^3}$</td>
<td>0</td>
</tr>
<tr>
<td>exponential decay</td>
<td>$e^{-x}, (1/3)^x$</td>
<td>0</td>
</tr>
</tbody>
</table>

If I have two functions $f, g$ that go to infinity I can compare how fast they go to infinity by looking at

$$\lim_{x \to \infty} \left| \frac{f}{g} \right| = \lim_{x \to \infty} \left| f \cdot \frac{1}{g} \right|$$

In a sense we are racing $f, g$ against each other; $f$ is getting big and $\frac{1}{g}$ is getting small.

e^x$ beats any polynomial in fact using L’Hospital we see that

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \lim_{x \to \infty} \frac{e^x}{n!} = \cdots = \lim_{x \to \infty} \frac{e^x}{n!} = \infty$$

This means in terms of growth rate $e^x$ is bigger than any polynomial. We will denote this $e^x \succ x^n$

**Definition 1.** Suppose $f, g$ are continuous functions defined on $[a, \infty)$ then

- if $\lim_{x \to \infty} \left| \frac{f}{g} \right| = \infty$ then $f$ dominates $g$
  $$f \succ g$$
- if $\lim_{x \to \infty} \left| \frac{f}{g} \right| = c > 0$ then $f$ and $g$ are asymptotic
  $$f \asymp g$$
- if $\lim_{x \to \infty} \left| \frac{f}{g} \right| = 0$ then $f$ is dominated by $g$
  $$f \ll g$$

We write $f \asymp g$ if $f \asymp g$ or $f \asymp g$.

We can use this for integrals:

**Fact:** If $f \succ g$ and we can show $\int_a^\infty f \, dx$ converges then we know $\int_a^\infty g \, dx$ converges and if we can show that $\int_a^\infty g \, dx$ diverges then we know that $\int_a^\infty f \, dx$ diverges.
Example 2. Does $\int_{1}^{\infty} e^{-x^2} \, dx$ converge?

Let’s compare with $e^{-x}$

$$\lim_{x \to \infty} \left| \frac{e^{-x}}{e^{-x^2}} \right| = \lim_{x \to \infty} \left| e^{x - x^2} \right| = \lim_{x \to \infty} e^{x^2 - x} = \infty$$

therefore $e^{-x} \succ e^{-x^2}$ and we can show that $\int_{1}^{\infty} e^{-x} \, dx$ converges so $\int_{1}^{\infty} e^{-x^2} \, dx$ converges.