Improper Integrals for Comparison

\[ \int_{1}^{\infty} \frac{1}{x^p} \, dx = \begin{cases} \infty & \text{if } p \leq 1 \\ \frac{1}{p-1} & \text{if } p > 1 \end{cases} \]

**Proof:**

\[ \int_{1}^{\infty} \frac{1}{x^p} \, dx = \begin{cases} \ln x |_{1}^{\infty} & \text{if } p = 1 \\ -x^{-p+1} |_{1}^{\infty} & \text{if } p \neq 1 \end{cases} + C \]

So

\[ \int_{1}^{\infty} = \lim_{b \to \infty} \left[ \frac{b^{1-p}}{1-p} - \frac{1}{1-p} \right] \]

or

\[ \lim_{b \to \infty} \left[ \ln b |_{1}^{\infty} \right] = \infty \]

\[ = \begin{cases} \infty & \text{if } 1-p > 0 \iff p < 1 \\ 0 - \frac{1}{1-p} = \frac{1}{p-1} & \text{if } p > 1 \end{cases} \]

\[ \int_{0}^{\infty} r^{-x} \, dx = \lim_{b \to \infty} \left[ -\frac{r^{-x}}{\ln(r)} \right] _{0}^{b} \]

\[ = \begin{cases} \frac{1}{\ln(r)} & \text{if } p > 1 \\ \infty & \text{if } 0 < r \leq 1 \end{cases} \]

**Simple Type II:** \( \int_{a}^{b} f(x) \, dx \) where \( f \) has an asymptote at \( a \) or at \( b \)

**Example:**

\[ \int_{0}^{\frac{1}{2}} \frac{dx}{\sqrt{x}} \]

\[ = \lim_{a \to 0^+} \int_{a}^{\frac{1}{2}} \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \left[ 2\sqrt{x} \right] _{a}^{\frac{1}{2}} \]

\[ = \lim_{a \to 0^+} \left[ 2\sqrt{\frac{1}{2}} - 2\sqrt{a} \right] \]

\[ = \frac{2}{2} \]

**Example:**

\[ \int_{0}^{\frac{1}{2}} \frac{dx}{(2x-1)^2} \]

\[ = \lim_{b \to \frac{1}{2}^-} \int_{0}^{b} \frac{dx}{(2x-1)^2} = \lim_{b \to \frac{1}{2}^-} \left[ \frac{1}{2} \cdot \frac{-1}{2x-1} \right] _{0}^{b} \]

\[ = \lim_{b \to \frac{1}{2}^-} \left[ -\frac{1}{4(2b-1)} + \frac{1}{4(-1)} \right] \]

\[ = -\infty \quad (\text{diverges}) \]
\[
\int \frac{1}{x^p} \, dx = \begin{cases} 
\ln |x| & \text{if } p = 1 \\
\frac{x^{1-p}}{1-p} & \text{if } p \neq 1 
\end{cases} + C
\]

\[
\int_1^\infty \frac{1}{x^p} \, dx = \begin{cases} 
\int \frac{1}{x} \, dx & \text{if } p = 1 \\
\frac{x^{1-p}}{1-p} & \text{if } p \neq 1 
\end{cases} + C
\]

\[
\int_0^\infty \frac{1}{\sqrt{x} \ (x+1)} \, dx = \text{two problems!}
\]

\[
= \int_0^1 + \int_1^\infty
\]

\[
\text{simple! only one problem each, and at an endpt.}
\]

\[
\int_0^{\infty} \frac{1}{\sqrt{x} \ (x+1)} \, dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x} \ (\sqrt{x} + 1)} \, dx = 2 \int \frac{du}{u^2 + 1} = 2 \arctan (\sqrt{x})
\]

\[
= 2 \left[ \arctan (1) - \arctan (e) \right] = 2 \left( \frac{\pi}{4} - 0 \right) = \frac{\pi}{2}
\]

\[
\sum \int_0^\infty \frac{1}{\sqrt{x} \ (x+1)} \, dx \text{ converges.}
\]