Lecture 6

Does $\int_{a}^{b} f(x) \, dx$ converge or diverge?

Ex Does $\int_{0}^{\infty} \frac{4x}{(x+1)^5} \, dx$ conv. or div.?

Integrand $\propto \frac{1}{x^2}$ as $x \to \infty$, so

$\int_{1}^{\infty} \frac{4x}{(x+1)^5} \, dx$ converges (since $\int_{1}^{\infty} \frac{1}{x^2} \, dx$ does)

and $\int_{0}^{1} \frac{4x}{(x+1)^5} \, dx$ is proper, so

$\int_{0}^{\infty} \frac{4x}{(x+1)^5} \, dx$ converges.

Rule: Convergence is unaffected by shifting endpoints (as long as no new issues are introduced.)

E.g.: $\int_{-1}^{\infty} \frac{4x}{(x+1)^5} \, dx$ ???

(new problem, at $x = -1$, integrand has an asymptote.)

So far, mainly been considering improper integrals that are "simple Type I": only one issue: an infinite endpoint.

$a$ and $b$ are real

"simple Type II": $f(x)$ has an asymptote at $x = a$ or $x = b$ (but not both)

Ex $\int_{0}^{1} \frac{1}{x} \, dx$

1. $= \lim_{a \to 0^+} \int_{a}^{1} \frac{1}{x} \, dx$ [write as limit]

2. $= \lim_{a \to 0^+} [\ln(1) - \ln(a)]$ [eval. $\int$]

3. $= +\infty$ [eval. limit]

Diverges

Ex $\int_{0}^{1} \frac{1}{\sqrt{x}} \, dx = \lim_{a \to 0^+} \int_{a}^{1} \frac{1}{\sqrt{x}} \, dx$

$= \lim_{a \to 0^+} [2\sqrt{1} - 2\sqrt{a}]$

$= 2$

Converges!
Integrals with multiple issues:

- Break them up into simple improper $\int$s.
- Converge only if all of the simple ones converge.
- Diverge if even one of the simple ones diverges.

Example:

$$\int_0^6 \frac{1}{(x-4)^{2/3}} \, dx$$

$$= \int_0^4 + \int_4^6$$

$$\uparrow$$

Evaluate separately.

$$\int_0^4 \frac{1}{(x-4)^{2/3}} \, dx = \lim_{b \to 4^-} \left[ -3(x-4)^{1/3} \right]_0^b$$

$$= 3 \cdot 4^{1/3}$$

$$= 3.4^{1/3}$$

Example:

$$\int_0^6 \frac{1}{(x-4)^{2/3}} \, dx = 3.2^{1/3}.$$