1 Lecture 6

A little review. Improper integrals

\[ \text{type I} \quad \int_a^\infty f(x) \, dx \quad \text{no undefined values} \]
\[ \int_{-\infty}^{b} f(x) \, dx \]
\[ \int_{-\infty}^{\infty} f(x) \, dx \]

\[ \text{type II} \quad \int_a^b f(x) \, dx \quad f(x) \text{ undefined somewhere in } [a, b] \]

\[ \text{combo} \quad \int_a^\infty f(x) \, dx \quad f(x) \text{ has some undefined values} \]

- \[ \int_1^\infty \frac{1}{x^p} = \frac{1}{p-1} \text{ for } p > 1 \text{ and diverges for } p < 1 \]
- \[ \int_0^\infty r^{-r} \, dx = \frac{1}{\ln(r)} \text{ for } r > 1 \]
- (new, for type II): \[ \int_0^1 \frac{1}{x^p} \, dx = \frac{1}{1-p} \text{ for } p < 1 \text{ and diverges for } p \geq 1 \]

**Example 1.** Does \( \int_{-\infty}^{-3} \frac{1}{9x^2+4} \, dx \) converge?

Type I. Probably compare with \( \frac{1}{x^2} \) but this has an issue at \( x=0 \). So break up the integral

\[ \int_{-\infty}^{-3} \frac{1}{9x^2+4} \, dx = \int_{-\infty}^{-3} \frac{1}{9x^2+4} \, dx + \int_{1}^\infty \frac{1}{9x^2+4} \, dx \]

Draw a picture...

Convergence only depends on the “tail” so I can focus only on \( \int_{1}^\infty \frac{1}{9x^2+4} \, dx \) and now I don’t need to worry that \( \frac{1}{x^2} \) isn’t defined at \( x=0 \).

Now use asymptotic algebra:

\( 9x^2 + 4 \gg 9x^2 \gg x^2 \) so \( \frac{1}{9x^2+4} \sim \frac{1}{x^2} \) and \( \int_{1}^\infty \frac{1}{x^2} \, dx \) converges so \( \int_{-\infty}^{-3} \frac{1}{9x^2+4} \, dx \) converges.

**Example 2.** Does \( \int_0^3 \frac{1}{x-1} \, dx \) converge?

Draw a picture

Only problem value is at \( x=1 \) so we split it up

\[ \int_0^3 \frac{1}{x-1} \, dx = \int_0^1 \frac{1}{x-1} \, dx + \int_1^3 \frac{1}{x-1} \, dx \]
\[ = \lim_{b \to 1} \int_0^b \frac{1}{x-1} \, dx + \lim_{a \to 1} \int_a^3 \frac{1}{x-1} \, dx \]
\[ = \lim_{b \to 1} \ln |x-1| \big|_0^b + \lim_{a \to 1} \ln |x-1| \big|_a^3 \]
\[ = \lim_{b \to 1} \ln |b-1| + \lim_{a \to 1} (\ln(2) - \ln|a-1|) \]
neither \lim is finite so the whole thing diverges.

Notice if we hadn’t expressed this as a limit we would have gotten the wrong answer.

(picture of generic function that has issues)

The general strategy: note each kind of infinity and break up into integrals into “simple” improper integrals.

**Example 3.** \( \int_{-5}^{3} \frac{1}{x^{3/5}} \, dx \). Only issue is at \( x=0 \)

\[
\int_{-5}^{3} \frac{1}{x^{3/5}} \, dx = \int_{-5}^{0} \frac{1}{x^{3/5}} \, dx + \int_{0}^{3} \frac{1}{x^{3/5}} \, dx
\]

\[
= \lim_{b \to 0^-} \int_{-5}^{b} \frac{1}{x^{3/5}} \, dx + \lim_{a \to 0^+} \int_{a}^{3} \frac{1}{x^{3/5}} \, dx
\]

\[
= \lim_{b \to 0^-} \frac{5}{2} \left( b^{2/5} - (-5)^{2/5} \right) + \lim_{a \to 0^+} \frac{5}{2} \left( 3^{2/5} - a^{2/5} \right)
\]

\[
= -(25)^{1/5} + 9^{1/5} < 0
\]

**Remark 4.** We spent some type talking about asymptotic relations for type I integrals. We will NOT do this for type II integrals. We don’t have to worry about asymptotic comparisons for type II integrals.

**Remark 5.** If we break up an integral into pieces \( \int f \, dx = A + B + C \) and if at least one pieces diverges then we say the whole thing diverges.