1 Lecture 7

Here is how you should use these integrals

- \( \int_1^{\infty} \frac{1}{x^p} \, dx = \frac{1}{p-1} \) for \( p > 1 \) and diverges for \( p < 1 \)
- \( \int_0^{\infty} r^{-z} \, dx = \frac{1}{\ln(r)} \) for \( r > 1 \)\( \)
- \( \int_0^{1} \frac{1}{x^p} \, dx = \frac{1}{1-p} \) for \( p < 1 \) and diverges for \( p \geq 1 \)

For convergence/divergence problems I’ll suggest a template. Let me demonstrate on some previous examples

\[
\int_{-3}^{\infty} \frac{1}{9x^2+4} \, dx \quad \text{conv/div?} \\
\int_{-3}^{1} \frac{1}{9x^2+4} \, dx + \int_{1}^{\infty} \frac{1}{9x^2+4} \, dx
\]

template:

\(-3 \leftarrow \text{proper} \int \rightarrow 1 \leftarrow \text{Type I: conv} \rightarrow \infty \)

final answer: convergent

\[
\int_{-5}^{3} \frac{1}{x^{3/5}} \, dx = \int_{-5}^{0} \frac{1}{x^{3/5}} \, dx + \int_{0}^{3} \frac{1}{x^{3/5}} \, dx
\]

\(-5 \leftarrow \text{Type II: conv} \rightarrow 0 \leftarrow \text{Type II: conv} \rightarrow 3 \)

final answer: convergent

and now something new: Direct comparison

Here is the idea:

Say we want to integrate \( \int_a^b p(x)q(x) \, dx \) and \( q(x) \) is continuous, bounded and positive on \([a,b]\) and \( p(x) \) has an asymptote at \( x=a \) but otherwise has no problems. And suppose \( p(x) \) is simple enough that we know how to integrate it

Then with a little work we can find constants so that \( m \leq q(x) \leq M \) then

\[
\Rightarrow \quad m \leq p(x) \leq p(x)q(x) \leq p(x)M \\
\Rightarrow \quad \int_a^b p(x) \, m \, dx \leq \int_a^b p(x)q(x) \, dx \leq \int_a^b p(x)M \, dx
\]

If both the upper bound and lower bound are finite then the integral we want \( \int_a^b p(x)q(x) \, dx \) converges. If the lower bound diverges then \( \int_a^b p(x)q(x) \, dx \)

Examples
\[ \int_0^4 \frac{1}{\sqrt{3x^3 + 4x^2}} \, dx \]
\[ \frac{1}{x} \frac{1}{\sqrt{3x + 4}} \]
\[ 0 \leq x \leq 4 \]
\[ 4 \leq 3x + 4 \leq 16 \]
\[ \frac{1}{4} \leq \frac{1}{\sqrt{3x + 4}} \leq \frac{1}{2} \]
\[ \frac{1}{4x} \leq \frac{1}{x\sqrt{3x + 4}} \leq \frac{1}{2x} \]

Integral diverges!

This one is slightly trickier

\[ \int_{-4}^{-1} \frac{1}{\sqrt{x(x-4)(x+4)}} \, dx \]
\[ \frac{1}{\sqrt{x+4}} \frac{1}{\sqrt{x^2-4x}} = pq \]
\[ -4 \leq x \leq -1 \]
\[ 1 \leq x^2 \leq 16 \]
\[ 4 \leq -4x \leq 16 \]
\[ 5 \leq x^2 - 4x \leq 32 \]
\[ \frac{1}{\sqrt{32}} \leq \frac{1}{\sqrt{x^2-4x}} \leq \frac{1}{\sqrt{5}} \]
\[ \int_{-4}^{\frac{1}{\sqrt{32\sqrt{x+4}}} \leq \int_{-4}^{\frac{1}{\sqrt{x+4}\sqrt{x^2-4x}} \leq \int_{-4}^{\frac{1}{\sqrt{5}\sqrt{x+4}}} \, dx \}

\[ \int_{x=-4}^{x=3} \frac{1}{\sqrt{x+4}} \, dx \]
\[ \int_{u=0}^{u=4} \frac{d(u+4)}{\sqrt{u}} \]
\[ \int_{u=0}^{u=3} \frac{d(u+4)}{\sqrt{u}} \]
\[ \int_{u=0}^{u=3} \frac{d(u+4)}{\sqrt{u}} \text{ converges} \]

Coming up with the bounds can take some getting used to. Often we can figure out these bounds because our \( q(x) \) will built from a trig function like \( \cos(x) \) and we can start with \( -1 \leq \cos(x) \leq 1 \) or, as in the examples we have worked out we can start with the limits of integration \( \int_a^b \) and try to build up to \( q(x) \). Generally, unless \( q(x) \) involves a trig function, it will be an increasing or decreasing function in which case the constants will be \( q(a) \) and \( q(b) \).
For example if $q(x) = \frac{1}{\sqrt{x^2 - 4}}$ then $q(-1) = \frac{1}{\sqrt{5}}$ and $q(-4) = \frac{1}{\sqrt{12}}$. 