Lecture 8

The basic idea of direct comparison is the same as limit comparison: If $f, g$ are positive functions and $0 \leq f \leq g$ on $[a, b]$ then

$$\text{if } \int_a^b f \, dx \text{ diverges then } \int_a^b g \, dx \text{ diverges}$$

$$\text{if } \int_a^b g \, dx \text{ converges then } \int_a^b f \, dx \text{ converges}$$

A special case of this is when we want determine if $\int_a^b pq \, dx$ converges or diverges and $p$ is undefined at some value and $q$ is a nice function. Then we can try to find bounds on $m \leq q \leq M$ so we can then get the bound $mp \leq pq \leq Mp$ and integrate and use direct comparison.

Tricky part is the constants but the most important part is the factorization $pq$. If you struggle to find $m \leq q \leq M$ then as a last resort say `'q is bounded by some unknown constants and because $\int_a^b p \, dx$ (converges/diverges)...'"

Most functions will be monotonic or trigonometric

In these examples lets first just determine their type and what methods we should probably use.

Example 1. $\int_1^\infty \frac{x+1}{\sqrt{x^4-x}} \, dx$ this is Type I and Type II

we can probably use limit comparison for the tail and direct comparison near $x = 1$

$$\int_1^\infty \frac{x+1}{\sqrt{x^4-x}} \, dx = \int_1^2 \frac{x+1}{\sqrt{x^4-x}} \, dx + \int_2^\infty \frac{x+1}{\sqrt{x^4-x}} \, dx$$

$$x^4-x=x(x^3-1)=x(x-1)(x^2+x+1)$$

$$\frac{x+1}{\sqrt{x^4-x}} = \frac{1}{\sqrt{x-1}} \frac{x+1}{x(x^2+x+1)}$$

need bounds for $\frac{x+1}{\sqrt{x(x^2+x+1)}}$ on $[1, 2]$

without getting into a messy calculation we see

$$r(x) = \frac{1}{\sqrt{x(x^2+x+1)}}$$

is decreasing on $[1, 2]$

$$2 \leq x+1 \leq 3$$

so $2r(2) \leq q(x) \leq 3r(1)$

$$\int_1^2 \frac{1}{\sqrt{x-1}} \, dx = \int_0^1 \frac{du}{u^{1/2}} \text{ which converges}$$

Put answer together in number line

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Example 2. \( \int_0^\pi \frac{\sin^2 x}{\sqrt{x}} \, dx \) this is II only. we can use direct comparison \( 0 \leq \sin^2(x) \leq 1 \).

Example 3. \( \int_0^\infty \frac{dx}{\sqrt{x(x+1)}} \) this a combo problem.
we can probably use limit comparison for the tail and direct comparison near \( x=0 \). Note \( x=-1 \) is outside the region of integration.

Example 4. \( \int_0^\infty \frac{1-\cos(x)}{x^2+1} \, dx \) This is I only but we probably need to use direct comparison.
\( 0 \leq 1-\cos(x) \leq 2 \)

\[
0 \leq \int_0^\infty \frac{1-\cos(x)}{x^2+1} \, dx \leq \int_0^\infty \frac{2}{x^2+1} \, dx
\]

\[
\int_0^\infty \frac{2}{x^2+1} \, dx \text{ converges by limit comparison}
\]
so \( \int_0^\infty \frac{1-\cos(x)}{x^2+1} \, dx \) converges

Sequences and Series!

Recall \((1-x)(1+x+\cdots+x^n) = (1-x^{n+1})\)

Multiply by \(a\)
\((1-x)(a + a x + \cdots + a x^n) = a (1-x^{n+1})\)

take the limit as \(n \to \infty\)

\[
1+x+\cdots+x^n = \frac{1-x^{n+1}}{1-x} \quad \text{works for any } x!
\]

\[
\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad -1 < x < 1
\]

\[
\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x} \quad -1 < x < 1
\]
But what if we want to know $4x^3 + \ldots + 4x^7 = 4x^3(1 + x + \ldots + x^4) = \frac{4(x^3 - x^9)}{1 - x}$

In particular

$$\sum_{n=a}^{b} c_n x^n = \frac{c(x^a - x^{b+1})}{1 - x}$$

**Example 5.** What is $12 + 24 + 48 + 96 + 192$? We see the ratio between successive terms is 2

$12 = 2^2 \times 3$ and $192 = 3 \times 64 = 2^6 \times 3$

$$12 + 24 + 48 + 96 + 192 = 3(2^2 + \ldots + 2^6) = \frac{3(2^2 - 2^7)}{-1} = -12(1 - 32) = 12 \times 31 = 372$$

Another application is turning a repeated fraction into a decimal. Consider $c = 0.81441441444 \ldots$

Then

$$c = 0.8 + 0.0144 + 0.0000144 + \ldots$$

$$= \frac{4}{5} + \frac{144}{10^4} + \frac{144}{10^7} + \ldots$$

$$= \frac{4}{5} + \frac{144}{10^4} \left(1 + \frac{1 + \left(\frac{1}{10^4}\right)^2 + \ldots}{1 + \left(\frac{1}{10^4}\right)^2 + \ldots}\right)$$

$$= \frac{4}{5} + \frac{144}{10^4} \left(\frac{1}{1 - \frac{1}{10^4}}\right)$$

$$= \frac{4}{5} \frac{144}{10^4} \left(\frac{1}{10^4 - 1}\right)$$

$$= \frac{4}{5} \frac{144}{10} \left(\frac{1}{999}\right)$$

$$\frac{4}{5} \frac{72}{5 \times 999} = \frac{999 \times 4 + 72}{5 \times 999} = \frac{4068}{4995}$$