For each of the following examples, give 5-10 minutes for them to work on their own, then present the solution. You don't have to do all of them. Please encourage them to compare answers with their neighbors and ask each other for help if they are having trouble getting started.

Example 1: Split $\int_0^1 \frac{1}{(x-1)(x+2)(x-3)} \, dx$ into a sum of simple improper integrals.

Answer: Reasons the integral is improper:
1. Infinite endpoint
2. Vertical asymptotes at 1, 3
(Note: vertical asymptote at -2 outside domain of integration, thus doesn't matter)

So, $\int_0^1 = \int_0^1 + \int_1^2 + \int_2^3 + \int_3^4 + \int_4^\infty$

Note: Cannot do $\int_0^1 + \int_2^3$

Because $\int_2^3$ has vertical asymptote at both endpoints.

Note: Choice of 2 as a splitting point between 1 and 3 was arbitrary — could split at e.g. 1.5, 2.5, etc.
Same with 4 as splitting point between 3 and $\infty$.

Picture:
Example 2: Compute $\int_0^1 \ln(x) \, dx$

Answer: Simple Type II; vertical asymptote at 0.

So, write as $\lim_{a \to 0^+} \int_a^1 \ln(x) \, dx$

$$= \lim_{a \to 0^+} \left( \ln(x) \right|_a^1 
= \lim_{a \to 0^+} (\ln(1) - \ln(a)) - (\ln(a) - a) 
= \lim_{a \to 0^+} 0 - 1 - \ln(a) - a 
= -1 - \lim_{a \to 0^+} \ln(a/a)
$$

Use L'Hopital:

$$\lim_{a \to 0^+} \ln(a) = \lim_{a \to 0^+} \ln(a/a) = \lim_{a \to 0^+} \frac{1/a}{1/a} = \lim_{a \to 0^+} -a = 0.$$ 

So $\int_0^1 \ln(x) \, dx = -1.$

Picture:
Example 3: \( \int_{-1}^{\infty} \frac{(e^x + 2)^{1/3}}{(e^x)^2 + e^x} \, dx \) converge or diverge?

Answer: Converges, Simple Type I, so use limit comparison.

\[
\frac{(e^x + 2)^{1/3}}{(e^x)^2 + e^x} \leq \frac{(e^x)^{1/3}}{(e^x)^2} = \frac{e^{1/3 \cdot x}}{e^{2x}} = e^{-5/2 \cdot x}
\]

Thus, \( \int_{-1}^{\infty} \frac{(e^x + 2)^{1/3}}{(e^x)^2 + e^x} \, dx \) converges by limit comparison with \( \int_{-1}^{\infty} e^{-5/2 \cdot x} \, dx \).

which converges because

\( \int_{-1}^{\infty} e^{-ax} \, dx \) converges for any \( a > 0 \).
Example 4  
\[ \int_0^{2\pi} \frac{2\sin \theta + 5}{N \theta} \, d\theta \]  

**Answer:** Converge  
Simple Type II (vertical asymptote at 0).  
So, can use direct comparison or try to evaluate.  
No obvious antiderivative, so direct comparison:  

Use "Bound the by stader":

\[
\frac{2\sin \theta + 5}{N \theta} = \frac{1}{N \theta} (2\sin \theta + 5) \\
\uparrow \\
\text{Problem causing asymptote} \\
\text{By stader}
\]

Need to bound \(2\sin \theta + 5\):

\[-1 \leq \sin \theta \leq 1\]
\[-2 \leq 2\sin \theta \leq 2\]
\[-2 + 5 \leq 2\sin \theta + 5 \leq 2 + 5 = 7\]
\[= 3\]
\[\Rightarrow \frac{2\sin \theta + 5}{N \theta} \leq \frac{7}{N \theta}\]

Thus \(\int_0^{2\pi} \frac{2\sin \theta + 5}{N \theta} \, d\theta\) converges because by direct comparison with \(\int_0^{2\pi} \frac{7}{N \theta} \, d\theta = \int_0^{2\pi} \frac{7}{\theta^{1/2}} \, d\theta\) which converges because \(\int_0^{2\pi} \frac{1}{\theta^{1/2}} \, d\theta\) converges for any \(p < 1\).
Example 5: \[ \int_0^\infty \frac{1}{\sqrt{x^5 + x^3}} \, dx \] converges or diverges?

Answer: Diverges

\[
\int_0^\infty \frac{1}{\sqrt{x^5 + x^3}} \, dx = \int_0^1 \frac{1}{\sqrt{x^5 + x^3}} \, dx + \int_1^\infty \frac{1}{\sqrt{x^5 + x^3}} \, dx
\]

\[
\int_0^1 \frac{1}{\sqrt{x^5 + x^3}} \, dx \quad \text{use direct comparison/bound the bystander.}
\]

\[
\frac{1}{\sqrt{x^5 + x^3}} = \frac{1}{\sqrt{x^3(x^2 + 1)}} = \frac{1}{x^{3/2}} \frac{1}{\sqrt{x^2 + 1}}
\]

Problem causing bystander

Bound \( \frac{1}{x^2 + 1} \) on \( 0 \leq x \leq 1 \):

\[
0 \leq x^2 \leq 1 \\
1 \leq x^2 + 1 \leq 2 \\
\frac{1}{2} \leq \frac{1}{x^2 + 1} \leq 1
\]

So

\[
\int_0^1 \frac{1}{x^{3/2}} \, dx \leq \int_0^1 \frac{1}{\sqrt{x^2 + 1}} \, dx
\]

Thus

\[
\int_0^1 \frac{1}{\sqrt{x^2 + 1}} \, dx \text{ diverges by direct comparison with}
\]

\[
\frac{1}{2} \int_0^{x_{3/2}} \frac{1}{x^{3/2}} \, dx \text{ which diverges since } \int_0^{1} \frac{1}{x^{3/2}} \, dx \text{ diverges for } p \geq 1.
\]

Since one simple part diverges, the whole thing diverges!

Note: \( \int_0^\infty \frac{1}{x^{3/2}} \, dx \) converges by limit comparison with \( \int_0^{1/2} \frac{1}{x^{3/2}} \, dx \), but didn't need to check this since first part diverged already!