Lecture 13

Recall we have this loose analogy

\[
\begin{align*}
\text{sequences} & \leftrightarrow \text{functions} \\
\text{partial sums} & \leftrightarrow \text{proper integrals} \\
\text{infinite series} & \leftrightarrow \text{improper integrals}
\end{align*}
\]

Here are some more aspects of this analogy

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } p > 1 \iff \int_1^{\infty} \frac{1}{x^p} \, dx \text{ converges } p > 1
\]

(p-series)

What about direct comparison and limit comparison of integrals? Do they have an analogy? The answer is yes!

**Theorem 1. (Comparison Test)** If \(0 \leq a_m \leq b_m\) for \(m\) sufficiently large then

- If \(\sum_{n=1}^{\infty} b_n\) converges then so does \(\sum_{n=1}^{\infty} a_n\)
- If \(\sum_{n=1}^{\infty} a_n\) diverges then so does \(\sum_{n=1}^{\infty} b_n\)

We also can do limit comparison with series.

**Example 2.** Does the series \(\sum_{n=1}^{\infty} \frac{7}{3n^2 + 3n + 1}\) converge?

Our intuition should tell us to try to compare with \(\sum_{n=1}^{\infty} \frac{7}{3n^2}\)

\[
3n^2 < 3n^2 + 3n + 1 \implies \frac{1}{3n^2 + 3n + 1} < \frac{1}{3n^2} \implies \frac{7}{3n^2 + 3n + 1} < \frac{7}{3n^2} \text{ for all } n \geq 1
\]

Therefore

\[
\sum_{n=1}^{\infty} \frac{7}{3n^2 + 3n + 1} \leq \sum_{n=1}^{\infty} \frac{7}{3n^2} \leq \frac{7}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

and the original series converges.

**Example 3.** (When comparison test fails)
Does $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converge or diverge?

Natural to try to use $2^n - 1 < 2^n$ so $\frac{1}{2^n} < \frac{1}{2^n-1}$, but this is useless! Why?

In this situation we can use limit comparison!

**Definition 4.** If $\{a_n\}$ and $\{b_n\}$ are positive sequences then

- $\{a_n\} \sim \{b_n\}$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$
- $\{a_n\} \prec \{b_n\}$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$
- $\{a_n\} \succ \{b_n\}$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$

**Theorem 5.** (Limit Comparison) If $\{a_n\}$, $\{b_n\}$ are positive sequences then

- If $\{a_n\} \sim \{b_n\}$ then $\sum a_n$ and $\sum b_n$ either both converge or both diverge
- If $\{a_n\} \prec \{b_n\}$ and $\sum a_n$ diverges then $\sum b_n$ diverges
- If $\{a_n\} \prec \{b_n\}$ and $\sum b_n$ converges then $\sum a_n$ converges

**Example 6.** (When comparison test fails)

$\{ \frac{1}{2^n-1} \} \prec \{ \frac{1}{2^n} \}$

and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges so $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges.

Let’s consider $\sum_{n=1}^{\infty} \frac{1}{n}$ but first let’s review $n$!

$$
\begin{align*}
n! &= n \times n - 1 \times n - 2 \times \cdots \times 2 \times 1 \\
0! &= 1 \\
1! &= 1 \\
2! &= 2 \\
3! &= 6 \\
4! &= 24 \\
5! &= 120 \\
6! &= 720 \\
7! &= 5040 \\
8! &= 40320 \\
9! &= 362880 \\
10! &= 3628800
\end{align*}
$$
Fact: 10! seconds is exactly 6 weeks!

the sequence \( \{n!\} \) grows really fast. In fact \( \{n!\} > \{r^n\} \) for every \( r > 1 \).

reason:

\[
\frac{r^n}{n!} = \frac{r}{n} \times \frac{r}{n-1} \times \ldots \times \frac{r}{2} \times \frac{r}{1}
\]

Once \( n > r \) the fraction will get smaller and smaller

\[
\frac{r^n}{n!} > \frac{r^{n+1}}{(n+1)!} > \ldots
\]

and eventually go to 0

**Example 7.** Does \( \sum_{n=1}^{\infty} \frac{1}{n!} \) converge?

We can do limit comparison with \( \frac{1}{2^n} \)

**Example 8.** Does \( \sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2 + n + 1} \) converge?

Use limit comparison

\[
\left\{ \frac{\sqrt{n+2}}{2n^2 + n + 1} \right\} \propto \left\{ \frac{\sqrt{n}}{2n^2} \right\} \approx \left\{ \frac{1}{2n^{1.5}} \right\}
\]

So \( \sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2 + n + 1} \) converges by limit comparison to the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^{1.5}} \)

**Example 9.** Does \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \) converge?

\[
n! = \frac{n}{n} \times \frac{n-1}{n} \times \frac{n-2}{n} \times \ldots \times \frac{1}{n} < \frac{2}{n^2}
\]

We can use direct comparison with \( \sum \frac{2}{n^2} \) to see the original series converges.

Note the expression about might make more sense are doing a specific example

\[
\frac{3!}{3^3} = \frac{3 \times 2 \times 1}{3 \times 3 \times 3}
\]

**Example 10.** \( \sum_{k=1}^{\infty} \frac{\ln k}{\sqrt[k]{e^n}} \)

This problem is a little tricky

\[
\ln k \propto \sqrt[k]{k}
\]

\[
\frac{\ln k}{\sqrt[k]{k}} \propto 1
\]

\[
\frac{\ln k}{\sqrt[k]{k} e^n} \propto \frac{1}{e^n}
\]

So \( \sum_{k=1}^{\infty} \frac{\ln k}{\sqrt[k]{k} e^n} \) converges by limit comparison with the geometric series \( \sum_{k=1}^{\infty} \frac{1}{e^n} \)