1 Lecture 15

Let’s try to see if the following series converges

\[ \sum (-1)^{n+1} \frac{1}{n^2} = 1 - 1/4 + 1/9 + \ldots. \]

We would like to compare it to \( \sum 1/n^2 \). Turns out we can do that

**Definition 1.** A series \( \sum a_n \) is absolutely convergent if \( \sum |a_n| \) converges.

**Theorem 2.** If a series is absolutely convergent then its convergent.

**Example 3.** \( \sum (-1)^{n+1} \frac{1}{n^2} \) is absolutely convergent so it converges.

**Example 4.** \( \sum (-1)^{n+1} \frac{1}{n} \) is not absolutely convergent so we don’t know if it converges

But we can answer this question using alternating series.

**Definition 5.** An alternating series is one of the form \( \sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - \ldots \) where each \( a_i \) is positive

\( \sum (-1)^{n+1} \frac{1}{n} \) is an alternating series

Let’s consider the following series

\[ \sum (-1)^{n+1} \frac{1}{n} \]

Notice that we have a decreasing sequences here:

\( \ldots < \frac{1}{3} < \frac{1}{2} < 1 \)

Let’s look at the partial sums

\[
\begin{align*}
S_1 &= 1 \\
S_2 &= \frac{1}{2} \\
S_3 &\approx 0.8333 \\
S_4 &\approx 0.5833 \\
S_5 &\approx 0.7833 \\
S_6 &\approx 0.6166 \\
\vdots &\vdots \\
S_{11} &\approx 0.7365 \\
S_{12} &\approx 0.6532
\end{align*}
\]
Here is the pattern we observe
\[
S_1 > S_3 > S_5 > \ldots \quad \left( -\frac{1}{2n} + \frac{1}{2n+1} \right) < 0
\]
\[
S_2 < S_4 < S_6 < \ldots \quad \left( -\frac{1}{2n-1} - \frac{1}{2n} \right) > 0
\]

We can draw a picture of this.

The sequences \( \{S_{2n}\} \) and \( \{S_{2n+1}\} \) are monotonic and bounded and they both converge. Also \( \{S_{2n+1} - S_{2n}\} = \left\{ \frac{1}{2n+1} \right\} \). So as \( n \to \infty \) both sequences converge to the same value.

**Theorem 6.** (Alternating Series Test) An alternating series \( \sum_{n=1}^{\infty} (-1)^{n+1}a_n \) converges if
\[
0 < a_{n+1} < a_n \text{ for all } n \text{ AND } \lim_{n \to \infty} a_n = 0
\]

We can get good estimates of alternating series using

**Theorem 7.** (Error Bounds for alternating series) If \( \sum_{n=1}^{\infty} (-1)^{n+1}a_n \) is an alternating series such that
\[
0 < a_{n+1} < a_n \text{ for all } n \text{ AND } \lim_{n \to \infty} a_n = 0
\]
and if \( S = \sum_{n=1}^{\infty} (-1)^{n+1}a_n \) then
\[
|S - S_n| < a_{n+1}.
\]

**Example 8.** Estimate \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \).

First we check the conditions of the alternating series test to see that it applies.

The partial sums are
\[
S_6 = 0.36806
\]
\[
\frac{1}{7!} < 0.0002
\]
So is \( S \) is the true sum then
\[
|S - S_6| < 0.0002
\]

**Example 9.** Does \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 1} \) converge?
The sequence \( \left\{ \frac{n^2}{n^3+1} \right\} \) is decreasing, look at \( f(x) = \frac{x^2}{x^3+1} \) and see that \( f'(x) = \frac{2x - x^2}{(x^3+1)^2} \).

Also

\[
\lim_{n \to \infty} \frac{n^2}{n^3+1} = 0
\]

So this series converges by alternating series test.

**Example 10.** \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}} \).

This is conditionally convergent.