Solutions for 20 series from Lecture 16 notes (Schaeffer)

a. \( \sum_{n=1}^{\infty} \sqrt{\frac{n}{n^4 + 3n}} \)

The series has \textit{algebraic} terms (polynomials, rational functions, and radicals, only), so the test to try is \textbf{limit comparison with a} \( p \)-\textbf{series}:

\[
\sqrt{\frac{n}{n^4 + 3n}} \approx \sqrt{\frac{n}{n^4}} = \frac{1}{n^{3/2}}
\]

so since \( \sum \frac{1}{n^{3/2}} \) converges (by the \( p \)-series test), our series converges as well. Because the series has only positive terms, it in fact converges \textbf{absolutely}.

b. \( \sum_{n=5}^{\infty} \left( -\frac{2}{e} \right)^{-n} \)

Several tests work here, but your first move should be to simplify the terms somewhat—notably, the negative power is somewhat annoying:

\[
\sum_{n=5}^{\infty} \left( -\frac{2}{e} \right)^{-n} = \sum_{n=5}^{\infty} \left( \frac{e}{2} \right)^n
\]

which is a \textbf{geometric series} with \( r = -e/2 \). Since \( e > 2 \), our \( r \) has absolute value \( \geq 1 \), so this geometric series diverges. You can also show this series diverges by the \textbf{divergence test} or by the \textbf{ratio test} (your \( L \) will come out to \( e/2 > 1 \)).

c. \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \)

This is a rare instance where you want to use the \textbf{integral test}. While generally we discourage using the integral test (favoring limit comparison, ratio, and alternating series most of the time), this series requires it.

The function \( f(x) = \frac{1}{x \ln x} \) is \textit{positive, decreasing, and continuous}, so we can apply the integral test. Our series therefore has the same convergence/divergence behavior as the integral

\[
\int_{2}^{\infty} \frac{dx}{x \ln x}
\]

To integrate this, we use \( u = \ln x \) and \( du = \frac{1}{x} \) \( dx \) to get \( \int \frac{dx}{x \ln x} = \ln \ln x + C \) so

\[
\int_{2}^{\infty} \frac{dx}{x \ln x} = \lim_{b \to \infty} [\ln \ln b - \ln \ln 2] = \infty
\]

and therefore the integral and the series both diverge.
d. \[ \sum_{n=2}^{\infty} \frac{(n+1)(n-1)}{4^n(n^2 - 1)} \]
Simplifying the terms of the series yields \( \sum \frac{1}{4^n} \), which is a geometric series with \( r = \frac{1}{4} \), so it converges (absolutely). In this case, it converges to \( \frac{1}{4^2} \cdot \frac{1}{3/4} = \frac{1}{12} \).

e. \[ \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \]
Series with logarithms either involve the integral test (with \( u = \ln x \), as in c.) or the limit comparison test, building on the fact that \( 1 < \ln n < n^p \) for any positive power \( p \).
Here, we want to use something like \( \ln n < \sqrt{n} \) (which is the statement above with \( p = 1/2 \)). Then \( \frac{\ln n}{n} < \frac{\sqrt{n}}{n} = \frac{1}{n^{3/2}} \).
Since \( \sum \frac{1}{n^{3/2}} \) converges by the \( p \)-series test, and \( \frac{1}{n^{3/2}} \) dominates the terms of our series asymptotically, our series converges as well. (It does so absolutely because its terms are all \( \geq 0 \).)

f. \[ \sum_{n=1}^{\infty} \frac{500000^n}{n!} \]
The exponential and factorial terms should alert you to use the ratio test:
\[
L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{500000^{n+1}}{(n+1)!} \cdot \frac{n!}{500000^n} \right| = \lim_{n \to \infty} \left| \frac{500000}{n+1} \right| = 0
\]
so since \( L < 1 \), the series converges absolutely.

g. \[ \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) \]
NOTE: You cannot split this up as \( \sum \frac{1}{n} - \sum \frac{1}{n+1} \) because these series do not converge, individually (see Theorem 9.2.1 in the book). Roughly speaking (and perhaps disturbing): The commutative property of addition \( a + b = b + a \) is not actually guaranteed when sums are infinite.
Instead: Simplify the terms. This yields
\[
\sum_{n=1}^{\infty} \left( \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} \right) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}
\]
which converges (absolutely) by \textbf{limit comparison with a p-series}, namely \( \sum \frac{1}{n^2} \).
(It actually converges to 1. If you write out the terms of the series you can see why!)

h. \[ \sum_{n=2}^{\infty} \frac{1}{n^2 \ln n} \]
Our two main options when faced with a series containing a logarithm is to try the \textbf{limit comparison test} using what we know about logarithmic growth, or the integral test. In this
case, we see that the \( \ln n \) term only makes the denominators of the terms grow more quickly than those of \( \frac{1}{n^2} \). That is,

\[
\frac{1}{n^2 \ln n} \prec \frac{1}{n^2}
\]

so since \( \sum \frac{1}{n^2} \) converges, our series also converges (absolutely).

i. \( \sum_{n=0}^{\infty} \frac{\sin n}{n!} \)

In addition to logs, another weird thing that pops up in series are sines and cosines. There are two cases: If the sine/cosine’s input contains \( \pi \), test values to see what the behavior is. Otherwise, i.e. if you just have \( \sin n \) or \( \cos n \), use the **absolute convergence test** and/or the **direct comparison test**, starting from \(-1 \leq \sin n, \cos n \leq 1\).

NOTE: You cannot apply direct comparison directly(!) to the series above, because it has *negative* as well as positive terms (the “fine print” of the test only allows terms that are \( \geq 0 \)).

ANOTHER NOTE: You cannot apply the alternating series test because the signs do not alternate every term.

Instead, note that

\[
\sum_{n=0}^{\infty} \left| \frac{\sin n}{n!} \right| = \sum_{n=0}^{\infty} \left| \frac{\sin n}{n!} \right|
\]

so since \( 0 \leq |\sin n| \leq 1 \), the \( n \)th term of the series above is between 0 and \( \frac{1}{n!} \). Since \( \sum \frac{1}{n!} \) converges (by the ratio test, for example), our series converges as well, and does so absolutely.

j. \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \)

This series converges by the **alternating series test**: The series is alternating and \( \frac{1}{\ln n} \) decreases and tends to zero as \( n \to \infty \).

The series in fact converges conditionally: Since \( \ln n \prec \sqrt{n} \), \( \frac{1}{\sqrt{n}} \prec \frac{1}{\ln n} \), so since \( \sum \frac{1}{\sqrt{n}} \) diverges, \( \sum \frac{1}{\ln n} \) diverges as well.

k. \( \sum \frac{n}{\sqrt{n} + 1} \)

Here the **divergence test** works, since the terms are \( \asymp n^{2/3} \), which tends to infinity as \( n \to \infty \). The series diverges.

l. \( \sum \frac{\sin(n \pi + \frac{\pi}{2})}{n^2 + 1} \)

Here you need to figure out what the sine term is doing, since its input contains \( \pi \): Writing
out the first few terms:

\[ \sum_{n=1}^{\infty} \frac{\sin(n\pi + \frac{\pi}{2})}{n^2 + 1} = \frac{\sin(\frac{\pi}{2})}{1} + \frac{\sin(3\pi)}{2} + \frac{\sin(\frac{5\pi}{2})}{5} + \frac{\sin(\frac{7\pi}{2})}{10} + \cdots \]

\[ = \frac{(+1)}{1} + \frac{(-1)}{2} + \frac{(+1)}{5} + \frac{(-1)}{10} + \cdots \]

\[ = 1 - \frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \cdots \]

so the series is alternating! Which is not obvious from how it’s written. The absolute values of the terms are \( \frac{1}{n^2 + 1} \) which is decreasing and tends to zero as \( n \to \infty \), so the series converges by the alternating series test.

However! It also converges absolutely: The absolute value series is just \( \sum \frac{1}{n^2 + 1} \), which converges by limit comparison with a \( p \)-series, for example.

m. \( \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} \)

Since \( 2n + 1 \) is always odd, \( (-1)^{2n+1} \) is always equal to \(-1\). This series is just \(-\sum \frac{1}{n} \), which is \(-1\) times the harmonic series, so it diverges.

n. \( \sum_{n=0}^{\infty} \frac{n! \cdot 2^n}{(2n)!} \)

Again, there are factorial and exponential terms, so this shouts “do the ratio test!”

Set up:

\[ L = \lim_{n \to \infty} \left| \frac{(n + 1)! \cdot 2^{n+1}}{(2n + 2)!} \cdot \frac{(2n)!}{n! \cdot 2^n} \right| \]

NOTE: \( (2(n + 1))! = (2n + 2)! \) NOT \( (2n + 1)! \). Use safety parentheses when changing \( n \) to \( (n + 1) \) in order to avoid mistakes.

Next, we cancel (like terms):

- Exponential canceling is pretty simple: \( \frac{2^{n+1}}{2^n} = 2 \).

- Factorial canceling is a bit trickier, and it helps to write them out if you’re having trouble:

\[ \frac{(n + 1)!}{n!} = \frac{1 \cdot 2 \cdot 3 \cdots n(n + 1)}{1 \cdot 2 \cdot 3 \cdots n} \]

so everything cancels except the \( (n + 1) \) upstairs: \( \frac{(n+1)!}{n!} = (n + 1) \).

- For \( \frac{(2n)!}{(2n+2)!} \),

\[ \frac{(2n)!}{(2n + 2)!} = \frac{1 \cdot 2 \cdot 3 \cdots (2n)}{1 \cdot 2 \cdot 3 \cdots (2n)(2n + 1)(2n + 2)} \]

so canceling leaves us with \( \frac{1}{(2n+1)(2n+2)} \).
After all the cancelation it remains to evaluate

\[ L = \lim_{n \to \infty} \left| \frac{2(n + 1)}{(2n + 1)(2n + 2)} \right| \]

which is equal to zero, because this is a rational function whose denominator has a larger degree than its numerator. Since \( L = 0 \), the series converges absolutely by the ratio test.

o. \( \sum_{n=1}^{\infty} \frac{\cos(2\pi n)}{n} \)

Again, the cosine contains \( \pi \) in its input, so it behooves us to figure out what it is doing:

\[
\sum_{n=1}^{\infty} \frac{\cos(2\pi n)}{n} = \frac{\cos(2\pi)}{1} + \frac{\cos(4\pi)}{2} + \frac{\cos(6\pi)}{3} + \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \cdots
\]

So this is actually just the harmonic series \( \sum \frac{1}{n} \), which we know diverges (by \( p \)-test or integral test, for example).

p. \( \sum_{n=1}^{\infty} (\arctan(1/n))^2 \)

We are given the hint that \( \arctan(\frac{1}{x}) < \frac{1}{x} \). Let’s verify this asymptotic relation!

\[
\lim_{x \to \infty} \left[ \frac{\arctan(1/x)}{1/x} \right] = \lim_{x \to \infty} \left[ \frac{\frac{1}{(1/x)^2+1} \cdot (-1/x^2)}{-1/x^2} \right] = \lim_{x \to \infty} \left[ \frac{1}{(1/x)^2 + 1} \right] = \lim_{x \to \infty} \left[ \frac{x^2}{1 + x^2} \right] = 1
\]

so they are asymptotic (above: first equality is L’Hôpital’s rule since the original limit is 0/0, second equality is canceling the \((-1/x^2)\)s, third equality is multiplying by \(x^2/x^2\)).

After we have the hint, the series is easy to figure out using the limit comparison test: The terms are asymptotic to \((1/n)^2\) and \(\sum \frac{1}{n^2}\) converges (by \( p \)-series test) and so our series converges as well (it does so absolutely, since all its terms are positive).

q. \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \)

Like c., this one requires the integral test (one of the two strategies we’ve mentioned that you should try when a logarithm is in play): Using the substitution \( u = \ln x \) and \( du = dx/x \),

\[
\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\ln x} + C
\]

So,

\[
\int_{2}^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{b \to \infty} \left[ -\frac{1}{\ln b} - \left( -\frac{1}{\ln 2} \right) \right] = \lim_{b \to \infty} \left[ \frac{1}{\ln 2} - \frac{1}{\ln b} \right] = \frac{1}{\ln 2}
\]

Since the integral above converges, our series converges (absolutely).
r. \( \sum_{n=1}^{\infty} \frac{e^{-n}}{n} \)

The **ratio test**, since there is an *exponential* term—this yields \( L = \frac{1}{e} \), so the series converges absolutely.

s. \( \sum_{n=1}^{\infty} \frac{(2n)!(3n)!}{(5n)!} \)

The **ratio test**, since the series has *factorials* in it.

Set up:

\[
L = \lim_{n \to \infty} \left| \frac{(2n+2)!(3n+3)!}{(5n+5)!} \cdot \frac{(5n)!}{(2n)!(3n)!} \right|
\]

Canceling like factorials as before,

\[
\frac{(2n+2)!}{(2n)!} = (2n+1)(2n+2) \\
\frac{(3n+3)!}{(3n)!} = (3n+1)(3n+2)(3n+3) \\
\frac{(5n)!}{(5n+5)!} = \frac{1}{(5n+1)(5n+2)(5n+3)(5n+4)(5n+5)}
\]

so,

\[
L = \lim_{n \to \infty} \left| \frac{(2n+1)(2n+2)(3n+1)(3n+2)(3n+3)}{(5n+1)(5n+2)(5n+3)(5n+4)(5n+5)} \right|
\]

How to take this limit? Should we expand everything? No! Since the numerator and denominator both have the same degree (which is 5), we only need to figure out what the *leading terms* will be after we expand:

\[
L = \lim_{n \to \infty} \left| \frac{(2n)^2(3n)^3 + (\cdots)}{(5n)^5 + (\cdots)} \right| = \lim_{n \to \infty} \left| \frac{108n^5 + (\cdots)}{15625n^5 + (\cdots)} \right|
\]

where the \((\cdots)\) represent terms with lower powers of \( n \)—these won’t matter in the limit. We get \( L = \frac{108}{15625} \), which is \(< 1\), so the series *converges absolutely*.

t. \( \sum_{n=1}^{\infty} \frac{2 + \sin n}{\sqrt{n}} \)

Since \( \pi \) is not inside the sine, this probably requires us to use the **direct comparison test** (or the absolute convergence test) with \(-1 \leq \sin n \leq 1\). The numerator is at most 3 and at least 1, so \( \frac{1}{\sqrt{n}} \leq \frac{2+\sin n}{\sqrt{n}} \leq \frac{3}{\sqrt{n}} \). Since \( \sum \frac{1}{\sqrt{n}} \) diverges (by *p*-series test), our series *diverges* as well.