1 Lecture 20

This is review for midterm 2. You should consult the Midterm 2 Expectations sheet on the course page; here are some highlights

Asymptotic Growth/Decay

\[ r^n \prec n! \prec n^n \]

Sequences and Series (Know the difference!)

Three types of sequences: bounded, monotonic, convergent. You should be able to come up with examples of all of these. Also

\[ \text{monotonic + bounded} \implies \text{convergent} \]

A series \( \sum a_n \) converges if the sequence \( \{ S_n \} \) or partial sums converges.

\[ \sum a_n \quad \text{series} \]
\[ \{ S_n \} \quad \text{partial sums} \]
\[ S_n = 1/n \]

Does the series converge?

\[ \sum b_n \quad \text{series} \]
\[ \{ T_n \} \quad \text{partial sums} \]
\[ \lim_{n \to \infty} b_n = 1/2 \]

Does the series converge?

Tests for convergence or divergence

an approximate guide for dealing with a series \( \sum a_n/b_n \)

given \( \frac{a}{b} \) check if \( a < b, \frac{a}{b} \to 0 \) or \( b < \frac{a}{b} \to \infty \) \( \implies \) divergence test

\[ \text{check if } a, b \text{ are algebraic} \implies \text{limit comp. to } p - \text{series} \]

\[ \text{check for } n! \text{ or } r^n \implies \text{ratio test} \]

\[ \text{check for } (-1)^n \implies \text{alt.ser.test} \]

\[ \text{check for } \cos(n), \sin(n) \implies \text{direct comp., abs.conv} \]

\[ \text{see a possible } u - \text{sub} \implies \text{integral test} \]
In regard to the integral test remember this inequality: if \( f \) is a positive, decreasing function then

\[
\int_1^\infty f(x) \, dx \leq \sum_{n=1}^{\infty} f(n)
\]

Drawing the picture for \( f(x) = 1/x \) is one way to remember this inequality.

**Example 1.** \( \sum_{n=1}^{\infty} \frac{(-2)^n}{n!} \)

we can apply the ratio test or the alternating test

\[
\lim_{n \to \infty} \left| \frac{\frac{(2)^{n+1} \cdot n!}{(n+1)! \cdot (2)^n}}{\frac{1}{n+1}} \right| = \lim_{n \to \infty} 2 \cdot \frac{1}{n+1} = 0
\]

or AltSeries test

\[
2^n - n! \text{ so } 0 < \frac{2^{n+1}}{(n+1)!} < \frac{2^n}{n!} \text{ and } \lim_{n \to \infty} \frac{2^n}{n!} = 0
\]

**Power Series**

\[ p(x) = \sum c_n(x - a)^n \]

The center of the power series is \( x = a \).

To find the IOC: apply the ratio test and check the endpoints!

The IOC always contains \( x = a \) look like one these:

\[
(-R + a, R + a) \\
(-R + a, R + a] \\
[-R + a, R + a) \\
[-R + a, R + a]
\]

**Example 2.** Suppose \( p(x) = \sum c_n(x - 7)^n \) is a series with some unknown IOC but we do know that

\[
p(5) \text{ converges} \\
p(11) \text{ diverges}
\]
what can we say about the following series?

- \( p(8) \) always converges
- \( p(4) \) diverge/converge
- \( p(9) \) diverge/converge
- \( p(2) \) always diverges

IOC could be as small as \([5, 9)\) or it could be as big as \([3, 11)\)

Here are power series you should know with their IOC!

<table>
<thead>
<tr>
<th>IOC</th>
<th>( n^{\text{th}} )-term</th>
<th>( \sum n ) notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{1-x} ) = 1 + x + x^2 + \cdots \quad (-1, 1) \</td>
<td>( x^n )</td>
<td>( \sum_{n=0}^{\infty} x^n )</td>
</tr>
<tr>
<td>( e^x ) = 1 + x + \frac{x^2}{2!} \quad (-\infty, \infty) \</td>
<td>( \frac{x^n}{n!} )</td>
<td>( \sum_{n=0}^{\infty} \frac{x^n}{n!} )</td>
</tr>
<tr>
<td>( \ln(1+x) = \frac{x}{1} + \frac{x^2}{3} \cdots \quad (-1, 1) \</td>
<td>( (-1)^{n+1} \frac{x^n}{n} )</td>
<td>( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} )</td>
</tr>
<tr>
<td>( \tan^{-1}(x) = \frac{x^3}{3} + \frac{x^5}{5} \cdots \quad [-1, 1] \</td>
<td>( (-1)^{n+1} \frac{x^{2n+1}}{2n+1} )</td>
<td>( \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1} )</td>
</tr>
</tbody>
</table>

write them down at the beginning of the exam so you can refer back to them later

**Example 3.** Find a power series for \( \frac{-3x}{3x-1} \)

\[
\frac{-3x}{3x-1} = \frac{3x}{1-3x} = 3x \frac{1}{1-3x} = 3x(1+3x+(3x)^2+\cdots) = 3x + (3x)^2 + (3x)^3 + \cdots = \sum_{n=1}^{\infty} 3^n x^n
\]
Let’s calculate it’s IOC. \( a_n = 3^n x^n \) so

\[
\frac{|a_{n+1}|}{a_n} = \frac{|3^{n+1} x^{n+1}|}{3^n x^n} = 3|x|
\]

\[
L(x) = \lim_{n \to \infty} 3|x| = 3|x|
\]

The end points are where \( 3|x| = 1 \) which is \( \pm 1/3 \). The series diverges in both cases.

So IOC = \((-1/3, 1/3)\)