Math 21 - 11/14

Today: Review of power series

Series they should have memorized:

<table>
<thead>
<tr>
<th>Function</th>
<th>Series</th>
<th>Center</th>
<th>IOC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1-x)</td>
<td>(\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots)</td>
<td>0</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>(e^x)</td>
<td>(\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots)</td>
<td>0</td>
<td>(-\infty, 0)</td>
</tr>
<tr>
<td>(\ln(1+x))</td>
<td>(\ln(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} )</td>
<td>0</td>
<td>((-1, 1])</td>
</tr>
<tr>
<td>(\ln(x))</td>
<td>(\ln(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x-1)^{k+1}}{k+1} )</td>
<td>1</td>
<td>((0, 2])</td>
</tr>
<tr>
<td>(\arctan(x))</td>
<td>(\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} )</td>
<td>0</td>
<td>([-1, 1])</td>
</tr>
</tbody>
</table>

Interval of convergence (they'll know this abbreviation)

Example: (As on Monday, give some time for them to work on)

Use the power series above to find series for the following, or explain why you cannot:

(a) \(\frac{1}{0.97}\)  (b) \(\ln(4)\)  (c) \(e^4\)  (d) \(\arctan(1)\)

Solution

(a): \(\frac{1}{0.97} = \frac{1}{1 - 0.03} = 1 + 0.03 + (0.03)^2 + (0.03)^3 + \cdots = 1 + \frac{3}{100} + \frac{3^2}{10^4} + \cdots\)

(b) Cannot use series above directly because \(4\) is not in the IOC for \(\ln(x)\).

(Note: Some students may write \(\ln(4) = -\ln(\frac{1}{4})\) then use series. This is great! But ask if they just stopped after saying it doesn't work because \(4\) is not in IOC.

(c) \(e^4 = 1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \cdots\)

(d) \(\pi = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\)
Finding the interval of convergence

Step 1: Use ratio test to identify endpoints

Step 2: Check each endpoint for convergence

Example: (Give time for students to work)

Find the IOC for

(a) \( \sum_{n=0}^{\infty} 3^n a^2 (x - 5)^n \)

(b) \( \sum_{n=0}^{\infty} \frac{n!}{n^n} (x - 1)^n \)  
   (Find endpoints only)

(c) \( \sum_{n=0}^{\infty} n! x^n \)

Solutions:

(a) Compute L: \( \lim_{n \to \infty} \left| \frac{3^n (x+3)^n}{3^n (x-5)^n} \right| = \lim_{n \to \infty} 3 \left( \frac{x+3}{x-5} \right)^n \)

So \( L = 3 \) when \( 3|x-5| = 1 \), \( |x-5| = \frac{1}{3} \)

So \( x = 5 \pm \frac{1}{3} \), endpoints are \( \frac{14}{3}, \frac{16}{3} \).

At \( \frac{14}{3} \): get \( \sum_{n=0}^{\infty} 3^n a^2 (-\frac{1}{3})^n \) \( \sum_{n=0}^{\infty} (-1)^n n^2 \)

Diverges by divergence test (limit of terms \( \neq 0 \))

At \( \frac{16}{3} \): get \( \sum_{n=0}^{\infty} 3^n a^2 (\frac{1}{3})^n \) \( \sum_{n=0}^{\infty} n^2 \)

Diverges by divergence test

So IOC is \( \left( \frac{14}{3}, \frac{16}{3} \right) \).
\[ L = \lim_{n \to \infty} \frac{\frac{(n+1)^n}{n^n} (x-1)^{3n+1}}{n! / x^n (x-1)^n} \]

\[ = \lim_{n \to \infty} (n+1) \cdot \frac{n}{(n+1)^n} |x-1|^3 \]

\[ = \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n |x-1|^3 \]

\[ = |x-1|^3 \lim_{n \to \infty} \left( \frac{1}{1+\frac{1}{n}} \right)^n = |x-1|^3 \left( \frac{1}{\lim_{n \to \infty} \frac{1}{n}} \right) \]

\[ L = 1 \text{ when } |x-1|^3 \left( \frac{1}{\lim_{n \to \infty} \frac{1}{n}} \right) = 1 \]

\[ |x-1|^3 = e \]

\[ |x-1| = e^{1/3} \]

End points are \( 1 - e^{1/3}, 1 + e^{1/3} \)

\[ L = \lim_{n \to a} |\frac{(n+1)^n}{n^n} |x|^n \]

\[ = \lim_{n \to a} (n+1) |x| \]

\[ = +\infty \text{ unless } x = 0 \]

\[ \text{Power series always converges at center, so IOC is just a single point, the center!} \]

\[ \text{IOC} = \{ 0 \} = \{ 0, 0 \}. \]
Example (Give time for students to work)

Find a power series with IOC $(-1, 5)$.

Solution Note: center is 2 (distance 3 from both end points).

One possibility:

\[
\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{3^n (n+1)}
\]

Ratio test gives \(1 \cdot 2 \cdot 3 = 6\) for endpoints.

at \(-1\) get \(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}\) converges by alternating series test.

at \(5\) get \(\sum_{n=0}^{\infty} \frac{1}{n+1}\) diverges (harmonic series).

Can guess this series by trial and error — or note that it comes from

\[
\ln(y) = \sum_{n=0}^{\infty} \frac{(-1)^n (y-1)^{n+1}}{n+1}
\]

by substituting \(y = \frac{-x + 5}{3}\) IOC \((0, 2)\).

\[
(0 = \frac{-x + 5}{3} \rightarrow \frac{x}{3} = 5, \quad \frac{-x + 5}{3} \rightarrow \frac{x}{3} = -1)
\]

\[
7 = \frac{-x + 5}{3} \rightarrow x = -1
\]

\[
6 = -x + 5 \rightarrow x = -1
\]

\[
1 = -x
\]