Growth, decay, and stabilization

If \( f(x) \) is a function defined on an interval like \([a, \infty)\), the absolute value of that function \(|f(x)|\) will exhibit one of four broadly defined behaviors as \( x \to \infty \):

- It will **grow** to infinity: \( \lim_{x \to \infty} |f(x)| = \infty \).
  Examples: \( x^p \) if \( p > 0 \), \( a^x \) if \( a > 1 \), \( \ln(x) \).

- It will **decay** to zero: \( \lim_{x \to \infty} |f(x)| = 0 \).
  Examples: \( x^p \) if \( p < 0 \), \( a^x \) if \( 0 < a < 1 \).

- It will **stabilize** to a positive constant: \( \lim_{x \to \infty} |f(x)| = \) a positive real number.
  Examples: constant functions (except the zero function), \( \frac{x}{x+1} \), \( \arctan x \).

- The limit \( \lim_{x \to \infty} |f(x)| \) will not exist, e.g. because of oscillation or other reasons.
  Examples: \( \sin x \), \( \cos x \).

In what follows we will only be working with functions that exhibit one of the first three behaviors above. (This means we will generally avoid trigonometric functions in these notes.)

Comparing the growth and decay of functions

In may fields (mathematics, physics, statistics, computer science, and even economics), it is often important to compare “how quickly” two functions grow to \( \infty \) (or decay to 0) as \( x \to \infty \). This kind of analysis is referred to as **asymptotic analysis**, and a good working knowledge of it is extremely helpful in Math 21 and beyond.

Here are some examples to get us started:

- Both \( x^2 \) and \( e^x \) grow to infinity as \( x \to \infty \). However, you might already have some sense that \( x^2 \) approaches infinity “more slowly” than \( e^x \) does. To see this mathematically, we take the ratio of the functions’ values, \( x^2/e^x \), and observe that it decays to 0 rapidly:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x^2 )</th>
<th>( e^x )</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>100</td>
<td>( \approx 22026 )</td>
<td>0.005</td>
</tr>
<tr>
<td>20</td>
<td>400</td>
<td>( \approx 485165195 )</td>
<td>0.0000008</td>
</tr>
<tr>
<td>30</td>
<td>900</td>
<td>( \approx 10686474581524 )</td>
<td>( \approx 0 )</td>
</tr>
<tr>
<td>40</td>
<td>1600</td>
<td>( \approx 2.35 \times 10^{17} )</td>
<td>( \approx 0 )</td>
</tr>
<tr>
<td>50</td>
<td>2500</td>
<td>( \approx 5.18 \times 10^{21} )</td>
<td>( \approx 0 )</td>
</tr>
</tbody>
</table>
Even though both functions grow to infinity, \( x^2 \) exhibits \textit{power/polynomial growth} while \( e^x \) exhibits \textit{exponential growth}, and power growth is \textit{slower} than exponential growth.

We record this relationship between these two functions as \( x^2 \prec e^x \), or in words, “\( x^2 \) is dominated by \( e^x \).”

- Both \( \frac{1}{x} \) and \( \frac{1}{x^2} \) decay to zero as \( x \to \infty \). However, \( \frac{1}{x^2} \) approaches zero more quickly:

\[
\begin{array}{|c|c|c|c|}
\hline
x & \frac{1}{x} & \frac{1}{x^2} & \text{ratio} \\
\hline
10 & 0.1 & 0.01 & 10 \\
20 & 0.05 & 0.0025 & 20 \\
30 & 0.03 & 0.001 & 30 \\
40 & 0.025 & 0.000625 & 40 \\
50 & 0.02 & 0.0004 & 50 \\
\hline
\end{array}
\]

Note that the ratio \( \left( \frac{1}{x} \right) / \left( \frac{1}{x^2} \right) = x \) grows to infinity as \( x \to \infty \). This is because even though both \( \frac{1}{x} \) and \( \frac{1}{x^2} \) decay to zero, \( \frac{1}{x} \) decays more slowly.

We record this relationship between these two functions as \( 1/x \succ 1/x^2 \), or in words, “\( 1/x \) dominates \( 1/x^2 \).”

- \( x^4 + 5x - 3 \) and \( x^4 \) both grow to infinity, but they stay very close to one another as \( x \to \infty \). Even though these functions are not \textit{literally} equal, the linear \( 5x \) and constant \( -3 \) terms in the first function “aren’t as important” as the \( x^4 \) term is when trying to approximate \( x^4 + 5x - 3 \) for large values of \( x \):

\[
\begin{array}{|c|c|c|c|}
\hline
x & x^4 + 5x - 3 & x^4 & \text{ratio} \\
\hline
10 & 10047 & 10000 & 1.0047 \\
20 & 160097 & 160000 & 1.00060625 \\
30 & 810147 & 810000 & 1.0001814815 \\
40 & 2560197 & 2560000 & 1.0000769531 \\
50 & 6250247 & 6250000 & \approx 1 \\
\hline
\end{array}
\]

Since the ratio of the two functions approaches 1, if we were \textit{estimating} \( x^4 + 5x - 3 \) for a very large value of \( x \), we could say that it is \( \approx x^4 \).

- The functions \( 3x - 5 \) and \( 7x + 1 \) both exhibit \textit{linear growth}.

Since \( \lim_{x \to \infty} \left( \frac{3x - 5}{7x + 1} \right) = \frac{3}{7} \), we have \( 3x - 5 \approx \frac{3}{7} \cdot (7x + 1) \) for large values of \( x \).

\textbf{Asymptotic relations}

To formalize our observations above, we introduce the following \textit{asymptotic relations}:

- We say that \( f(x) \) and \( g(x) \) are \textit{asymptotic} to one another if \( \lim_{x \to \infty} \left( \frac{f(x)}{g(x)} \right) = \text{a constant } c \neq 0 \) and we write \( f(x) \prec g(x) \) [as \( x \to \infty \)] in this case.
The relationship \( f(x) \asymp g(x) \) means there is a nonzero constant \( c \) such that \( f(x) \approx cg(x) \) for large values of \( x \).

**Note:** In the special case that \( \lim_{x \to \infty} \left( \frac{f(x)}{g(x)} \right) = 1 \), we write \( f(x) \sim g(x) \). You do not need to know this for Math 21 but you may see this notation used elsewhere.

- We say that \( f(x) \) is dominated by \( g(x) \) if
  \[
  \lim_{x \to \infty} \left( \frac{f(x)}{g(x)} \right) = 0
  \]
  and we write \( f(x) \lessdot g(x) \) [as \( x \to \infty \)] in this case.

  If both \( f \) and \( g \) grow to infinity, then \( f \lessdot g \) if \( f \) grows more slowly than \( g \), and if both decay to zero, then \( f \lessdot g \) if \( f \) decays more quickly than \( g \).

- We say that \( f(x) \) dominates \( g(x) \) if
  \[
  \lim_{x \to \infty} \left( \frac{f(x)}{g(x)} \right) = \infty
  \]
  and we write \( f(x) \succ g(x) \) [as \( x \to \infty \)] in this case.

  If both \( f \) and \( g \) grow to infinity, then \( f \succ g \) if \( f \) grows more quickly than \( g \), and if both decay to zero, then \( f \succ g \) if \( f \) decays more slowly than \( g \).

For the examples on page 2...

- \( x^2 \lessdot e^x \) because \( e^x \) grows to infinity more quickly than \( x^2 \) does. Mathematically,
  \[
  \lim_{x \to \infty} \left( \frac{x^2}{e^x} \right) = \lim_{x \to \infty} \left( \frac{2x}{e^x} \right) = \lim_{x \to \infty} \left( \frac{2}{e^x} \right) = 0
  \]
  where the first two equalities follow from L’Hôpital’s rule. **Note:** No absolute value signs are necessary because \( x^2 \) and \( e^x \) are both nonnegative for all values of \( x \).

- \( \frac{1}{x} \succ \frac{1}{x^2} \) because \( \frac{1}{x} \) decays to zero more slowly than \( \frac{1}{x^2} \):
  \[
  \lim_{x \to \infty} \left( \frac{1/x}{1/x^2} \right) = \lim_{x \to \infty} \left( \frac{x^2}{x} \right) = \lim_{x \to \infty} (x) = \infty
  \]

- \( x^4 + 5x - 3 \succ x^4 \) because
  \[
  \lim_{x \to \infty} \left( \frac{x^4 + 5x - 3}{x^4} \right) = \lim_{x \to \infty} \left( \frac{1 + 5x^{-3} - 3x^{-4}}{1} \right) = \frac{1}{1} = 1
  \]
  where in the first equality we divided the numerator and denominator by \( x^4 \) and in the second we used \( x^{-3}, x^{-4} \to 0 \) as \( x \to \infty \).

- The limit in the fourth example on page 2 shows that \( 3x - 5 \simeq 7x + 1 \).
Here are some general rules about the $\succ$, $\prec$, and $\asymp$ relations:

- If $P(x)$ and $Q(x)$ are polynomials, then $P \asymp Q$ if they have the same degree*, $P \prec Q$ if the degree of $P$ is less than the degree of $Q$, and $P \succ Q$ if the degree of $P$ is greater than the degree of $Q$.

* The degree of a polynomial is the highest power of $x$ appearing in the polynomial.

- In particular, if $P(x)$ has degree $d$, then $P(x) \asymp x^d$.

- If $f_1 \asymp f_2$ and $g_1 \asymp g_2$, then $f_1 / g_1 \asymp f_2 / g_2$. Why?

$$
\lim_{x \to \infty} \left( \frac{f_1(x)}{g_1(x)} \right) = \lim_{x \to \infty} \left( \frac{f_1(x)}{f_2(x)} \cdot \frac{g_2(x)}{g_1(x)} \right) = \lim_{x \to \infty} \left( \frac{f_1(x)}{f_2(x)} \right) \cdot \lim_{x \to \infty} \left( \frac{g_2(x)}{g_1(x)} \right)
$$

Since both limits in the last expression are nonzero constants (because we know that $f_1 \asymp f_2$ and $g_1 \asymp g_2$), the whole thing is a nonzero constant!

- If $a > 1$, then $a^x \succ x^p$ for any $p > 0$.

  Exponential growth is always faster than power growth.

- If $0 < a < 1$, then $a^x \prec x^p$ for any $p < 0$.

  Exponential decay is always faster than power decay.

- If $p > 0$ then $x^p \succ \ln x$. Power growth is faster than logarithmic growth.

- If $p < q$ then $x^p \prec x^q$. Power growth/decay is ordered by exponent.

- If $0 < a < b$ then $a^x \prec b^x$. Exponential growth/decay is ordered by base.