Instructions

Read section 7.6 of the textbook and the notes on growth and decay (linked on the course website under Prerequisites, Notes, and Handouts).

Complete all book problems and Supplemental Problems A–C.

Unless otherwise specified, you will need to show your work to receive full credit.

Book Problems

• Section 7.6: 6*, 10*, 12*, 22*, 26*, 44

  * If the integral converges, find its exact value. You may need to use an integration table (there is one at the end of your book, but it might be missing a page due to a printer’s error—the full table can be found on the course website under Handouts).

  Note: Since a standard course on integral calculus (e.g. Math 20) is a prerequisite for this course, we are assuming you know how to evaluate integrals using substitution/change-of-variables (on exams, too!).

• Section 7.7: 2, 4, 6, 8

  Note: While the textbook uses the somewhat vague “behaves like” terminology, you should use the fact that if $f(x)$ and $g(x)$ are both defined and on $[a, \infty)$ and $f(x) \approx g(x)$, then $\int_a^\infty f(x) \, dx$ and $\int_a^\infty g(x) \, dx$ either both converge or both diverge.

  You should compare these integrals to ones in the “Useful Integrals for Comparison” box before Example 3 of 7.7—you do not have to re-prove that these useful integrals converge, but you should take a limit that establishes the necessary $\approx$ relation between integrands, showing your work.

Problem A: Integrating Rational Functions

A rational function is a function of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials (and $Q(x)$ is not equal to the constant function 0). For example,

$$\frac{1}{x}, \quad \frac{1}{4x^2 - 1}, \quad \frac{x^3}{1 - x}, \quad \frac{5x^2 + 3x + 1}{1}$$

are all rational functions. Of course, the last one is a polynomial, but any polynomial is also a rational function, with $Q(x) = 1$. 
It is a fun fact that every rational function can be integrated symbolically. This relies in part on an integration technique called \textit{partial fraction decomposition} (PFD), an algebraic calculation that takes a “complicated” rational function (that we do not know how to integrate) and re-expresses it as a sum of “simple” rational functions (that we do know how to integrate).

Because PFD requires a good deal of delicate algebra and at least two lectures to learn well, \textbf{you do not need to know how to compute partial fraction decompositions for this course}. If you ever need a PFD on an exam to evaluate an improper integral, the PFD will be given to you (as in the problems below). It is important, however, that you know that the method of partial fractions exists and what it is used for.

a. Verify by doing some algebra that

\[
\frac{1}{x^2 - 1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}
\]

The expression on the right is the partial fraction decomposition (PFD) of the rational function on the left. \textit{Hint: Start with the left-hand side.}

b. Evaluate \(\int \frac{dx}{x^2 - 1}\) using part (a).

To get the correct answer in part (c), your answer for this part should only contain one appearance of the \(\ln\) function. You may need to apply some log rules (there’s definitely a list of these somewhere in the textbook!).

c. Using (b), evaluate the improper integral \(\int_{1}^{\infty} \frac{dx}{x^2 - 1}\).

(If you believe the integral diverges, say so, and justify your answer.)

d. Evaluate \(\int \frac{x^2 - x - 1}{2x^4 - x^3 + 2x^2 - x} \, dx\) using the PFD

\[
\frac{x^2 - x - 1}{2x^4 - x^3 + 2x^2 - x} = \frac{1}{x^2 + 1} + \frac{1}{x} - \frac{2}{2x - 1}
\]

Express your answer using at most one \(\ln\) function (there may be other functions involved in your answer, but you should not have more than one \(\ln\)).

e. Using (d.), evaluate the improper integral \(\int_{1}^{\infty} \frac{x^2 - x - 1}{2x^4 - x^3 + 2x^2 - x} \, dx\)

(If you believe the integral diverges, say so, and justify your answer.)
Problem B: The $\prec$, $\simeq$, and $\succ$ relations

In class and on the notes on asymptotic analysis (available on the course website under Handouts), we defined the relations $\prec$ (is dominated by), $\simeq$ (is asymptotic to / behaves like), and $\succ$ (dominates) on functions that are “nice enough.”

These relations can be used to compare how quickly functions grow (tend to $\infty$) or decay (tend to $0$) asymptotically, as $x \to \infty$. They are particularly useful in computer science, where they are used to evaluate the efficiency of algorithms (if you’ve taken a CS course, these relations are related to “big-O” notation).

In problems (a–h) below, fill in the blank with the correct relation: $\prec$, $\simeq$, or $\succ$. If you are confident in your answer, you do not need to show your work (i.e., by taking limits of $\left| \frac{f(x)}{g(x)} \right|$). However, incorrect answers with some work attached might receive partial credit.

a. $x \underline{\mathbin{\text{_____}}} 2x^3 + 1$

b. $2^{-x} \underline{\mathbin{\text{_____}}} e^{-x}$

c. $x^2 \ln x \underline{\mathbin{\text{_____}}} 10^{256} \cdot x^2$

d. $\ln x \underline{\mathbin{\text{_____}}} \sqrt{x}$

e. $x \ln x \underline{\mathbin{\text{_____}}} (\ln x)^2$

f. $x \underline{\mathbin{\text{_____}}} \sqrt{4x^2 + 1}$

g. $\ln(x^2) \underline{\mathbin{\text{_____}}} \ln x$

h. $\frac{1}{x^2} \underline{\mathbin{\text{_____}}} \frac{x + 1}{x^5}$

Problem C: Rules for Asymptotic Growth Relations

In (a–c), decide whether the proposed rule about the relations $\prec$, $\simeq$, and/or $\succ$ is TRUE or FALSE, in general.

- If the statement is true, use limit rules to explain why. A list of limit rules is available on the course website (“limits cheat sheet”).

- If it is false, write down examples of functions $f(x)$ and $g(x)$ that demonstrate why.

We will assume that $f(x)$ and $g(x)$ are “nice enough.” Practically, this means no trig functions should be involved.

a. If $f(x) \prec g(x)$, then $f(x) + g(x) \simeq g(x)$.

b. If $f(x) \prec g(x)$, then $\frac{1}{f(x)} \succ \frac{1}{g(x)}$.

c. If $f(x) \prec g(x)$, then $\ln |f(x)| \prec \ln |g(x)|$. 