Math 21, Winter 2018 — Schaeffer/Wieczorek
Stanford University

Homework 1, due Friday, January 18th, at 10:00 AM, on Gradescope

Information and instructions

The contents of this assignment correspond roughly to the first 3–4 lectures.

The evaluation of improper integrals like those in 5–8 are covered in Section 7.6 of the textbook. The textbook does not adequately cover the material for questions 10–15; this material is covered in two handouts linked from the main course page and in lecture notes.

For full credit you must complete problems 1–15. This assignment is broken up into conceptual questions (which are vital for your understanding of the course material), routine questions (“textbook-style” problems). Later assignments will also include in-depth questions, which combine multiple concepts and/or give broader context (mathematical, historical, scientific, etc.) for the course material.

Conceptual questions

1. In the 2nd century BCE, the Greek mathematician Archimedes gave the formula

\[
\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{1}{3}
\]

where the sum on the left-hand side has infinitely many terms.

a. What is the pattern of the terms on the left-hand side of Archimedes’ formula? What do you think is the next term in the sum?

b. Archimedes justified this formula by observing that you can break up a $1 \times 1$ square into infinitely many smaller squares as in the picture below (the numbers indicate the side lengths of the squares that are shaded purple):

![Archimedes' square diagram]

c. Explain why the purple (darker) and two yellow (lighter) regions all have the same area, and why they must therefore each (individually) have total area $1/3$.

d. Explain why the area of the purple region can be expressed as the infinite sum on the left-hand side of Archimedes’ formula.
2. A *bounded* region of the plane is one that can be contained a rectangle of finite area. An *unbounded* region of the plane is one that is not bounded (it cannot be contained in any rectangle of finite area).

   a. Is it possible for a bounded region to have infinite area?  
      If so, give or describe an example. If not, explain why.
   
   b. Is it possible for an unbounded region to have finite area?  
      If so, give or describe an example. If not, explain why.

3. Below are six functions (A–F) $f(x)$ that are continuous (and defined) on $[1, \infty)$, and that satisfy $f(x) \geq 0$ for all $x \geq 1$:

   A. $f(x) = 0$  
   B. $f(x) = 1$  
   C. $f(x) = \frac{1}{x}$  
   D. $f(x) = \frac{1}{x^2}$  
   E. $f(x) = \frac{x}{x+1}$  
   F. $f(x) = e^{-x}$.

These all have simple antiderivatives, except E:  

$$
\int \frac{x}{x+1} \, dx = x - \ln |x+1| + C
$$

Below are five properties (i–v) a function defined on $[1, \infty)$ can have:

   i. $f(x) > 0$ for all $x \geq 1$.
   ii. There is $c > 0$ such that $f(x) \geq c$ for all $x \geq 1$. ("$f(x)$ is bounded away from zero.")
   iii. $\lim_{x \to \infty} f(x) = 0$. ("$f(x)$ decays as $x \to \infty$."")
   iv. $\int_1^\infty f(x) \, dx$ is finite (i.e. the integral converges).
   v. $\int_1^\infty f(x) \, dx$ is infinite (i.e. the integral diverges).

And now the problems:

a. Make a table with six rows (corresponding to functions A–F) and five columns (corresponding to properties i–v). It should look something like this:

<table>
<thead>
<tr>
<th>i.</th>
<th>ii.</th>
<th>iii.</th>
<th>iv.</th>
<th>v.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>B.</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>C.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D.</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>E.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Draw an X in each square of the table if the function in that row has the property in that column. (We’ve filled out the first row. You do not have to justify your answers.)
Using your answers above, in parts (b–d) below, decide whether the given statement is true or false for a nonnegative continuous function \( f(x) \) defined on \([1, \infty)\). You do not need to justify your answers.

b. If \( f(x) \) is bounded away from zero, then \( \int_1^\infty f(x) \, dx \) must diverge.

c. If \( f(x) \) decays as \( x \to \infty \), then \( \int_1^\infty f(x) \, dx \) must converge.

d. If \( \int_1^\infty f(x) \, dx \) converges, then \( f(x) \) must decay as \( x \to \infty \).

4. It is a fun fact that \( \int_1^\infty \frac{dx}{x^2+1} \) converges. In fact, \( \int_1^\infty \frac{dx}{x^2+1} = \frac{\pi}{4} \).

   a. Sketch a graph of \( y = \frac{1}{x^2+1} \) and shade the region whose area is measured by the integral.

   b. Remembering that \( 1 < \sqrt{3} \) explain why we know that \( \int_1^{\sqrt{3}} \frac{dx}{x^2+1} \) also converges, without having to compute the integral all over again. (It may help you to draw another picture, though this is not required.)

   c. Remembering that \( 0 < 1 \), explain why we know that \( \int_0^\infty \frac{dx}{x^2+1} \) also converges, without having to compute the integral all over again. (It may help you to draw another picture, though this is not required.)

   d. The integral \( \int_1^\infty \frac{dx}{x^2+x} \) also converges. Explain why the argument from (c) does not show that \( \int_0^\infty \frac{dx}{x^2+x} \) converges (it actually diverges, but the point here is to figure out why your previous argument fails).

**Routine problems**

5. Exercise 6 in Section 7.6 of the textbook (either edition).


7. Exercise 12 in Section 7.6 of the textbook (either edition).

8. The probability that a randomly chosen DeLux brand light bulb lasts between \( a \) hundred and \( b \) hundred hours is given by

   \[ \int_a^b 0.012e^{-0.012t} \, dt \]

   DeLux claims that 90% of their lightbulbs last 1000 hours or more. Is this statement accurate? Justify your answer.

9. A rational function is a function of the form \( \frac{P(x)}{Q(x)} \) where \( P(x) \) and \( Q(x) \) are polynomials. The integrands in problems 4, 5, and 7 were all rational functions.

   Every rational function can be integrated symbolically. This relies in part on an integration technique called **partial fraction decomposition** (PFD), an algebraic operation that takes a “complicated” rational function (that we do not know how to integrate) and re-expresses it as a sum of “simple” rational functions (that we do know how to integrate). For example,

   \[ \frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)} \]
Above, the original rational function (on the left) has been “broken apart” into simpler rational functions (on the right). The right-hand side is easy to integrate now, because
\[ R(A + B) = RA + RB. \]
This approach ultimately yields the integral formula
\[ \int \frac{dx}{x^2 - 1} = \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C. \]

Because PFD requires a good deal of delicate algebra and some time to learn well, you do not need to know how to compute partial fraction decompositions for this course. If you ever need a PFD on an exam to evaluate an improper integral, the PFD will be given to you (as below).

a. Evaluate the indefinite integral
\[ \int \frac{2x^2 + 8x - 5}{4x^4 + 8x^3 + 19x^2 + 32x + 12} \, dx \]
using the PFD
\[ \frac{2x^2 + 8x - 5}{4x^4 + 8x^3 + 19x^2 + 32x + 12} = \frac{1}{x^2 + 4} - \frac{1}{2x + 1} + \frac{1}{2x + 3} \]

Notes: To get the right answer in part (b), you will need to express your answer using at most one instance of \( \ln \). If you have two or more \( \ln \)s, combine them, using basic log rules! Also, since this is an indefinite integral, your answer should end with +C.

b. Evaluate the convergent improper integral
\[ \int_0^\infty \frac{2x^2 + 8x - 5}{4x^4 + 8x^3 + 19x^2 + 32x + 12} \, dx. \]

In 10–13, determine whether the integral converges or diverges by using limit comparison (this is what the book means by the vague language “behaves like”). Limit comparison and asymptotic relations are covered in two handouts linked from the course website.

10. Exercise 2 in Section 7.7 of the textbook (either edition).
11. Exercise 4 in Section 7.7 of the textbook (either edition).
12. Exercise 6 in Section 7.7 of the textbook (either edition).
13. Exercise 8 in Section 7.7 of the textbook (either edition).
14. In (a–f) below, determine which asymptotic relation symbol (\( \ll, \approx, \) or \( \gg \)) correctly fills in the blank. Justify your answer by evaluating an appropriate limit.

a. \( x^2 + 1 \ll 1 - 3x + 5x^2 \)
b. \( \ln x \approx x^{0.001} \)
c. \( e^x \gg 2^x \)
d. \( x^{-3} \ll 3^{-x} \)
e. \( x \gg x^{1000} \)
f. \( \ln x \ll \ln(x^{1000}) \)

15. Determine the (unique) value of \( p \) that makes \( x^p \approx \sqrt[4]{\frac{x}{x^4 + 1}} \) true.