Instructions

1. Read:
   (a) Section 7.7 of the textbook (either edition)

2. Complete:
   (a) Exercises 1-10 below
   (b) Problems D–F below

Unless otherwise specified, show your work to receive full credit.

Exercises

For Exercises 1-5, determine if the improper integral converges or diverges, and, if the integral converges, find its exact value. You may need to use an integration table (there is one at the end of your book, but it might be missing a page due to a printer’s error—the full table can be found on the course website under Handouts).

1. \[ \int_{-\infty}^{\infty} \frac{dz}{z^2 + 25} \]
2. \[ \int_{\pi/4}^{\pi/2} \frac{\sin x}{\cos x} \, dx \]
3. \[ \int_{0}^{1} x^4 + \frac{1}{x} \, dx \]
4. \[ \int_{0}^{2} \frac{1}{\sqrt{4 - x^2}} \, dx \]
5. \[ \int_{0}^{3} \frac{y \, dy}{\sqrt{9 - y^2}} \]

6. Explain what is wrong with the following statement:
   If \( \int_{1}^{\infty} f(x) \, dx \) diverges, then \( \lim_{x \to \infty} f(x) \neq 0. \)

(Exercises continued on next page)
For Exercises 7-9, decide if the improper integral converges or diverges.

**WARNING/REMEMBER:** Limit comparison (the “behaves like” principle) can only help you determine if an improper integral of the form \( \int_a^\infty f(x) \, dx \) (Type I) converges or diverges. You cannot use it to determine divergence and convergence if the integrand is unbounded (Type II). For example, asymptotic analysis can be applied to the integral \( \int_1^\infty \frac{dx}{1+x} \), but not to the integral \( \int_5^8 \frac{6}{\sqrt{t-5}} \). On the other hand, direct comparison (using inequalities like \( \leq \)) can be applied to both Type I and Type II integrals.

7. \[
\int_1^\infty \frac{dx}{x^3 + 1}
\]

8. \[
\int_0^1 \frac{d\theta}{\sqrt{\theta^4 + \theta}}
\]

9. \[
\int_1^\infty \frac{2 + \cos \phi}{\phi^2} \, d\phi.
\]

10. Explain what is wrong with the following statement:
    If \( 0 \leq f(x) \leq g(x) \) and \( \int_0^\infty g(x) \, dx \) diverges, then \( \int_0^\infty f(x) \, dx \) diverges.

**Problem D: Using geometry to relate improper integrals**

Recall the integration formula
\[
\int \ln x \, dx = x \ln x - x + C
\]
(which can either be memorized or derived using integration by parts).

a. Using the integration formula above, show that
\[
\int_0^1 \ln x \, dx
\]
converges, and find its value. *Note: You will probably need L’Hôpital’s rule.*

b. Give a *geometric* reason why
\[
\int_0^1 \ln x \, dx = -\int_0^\infty e^x \, dx
\]
then compute the value of the improper integral on the right, and use this to confirm your answer in part (a).

c. Give a *geometric* reason why
\[
\int_0^1 \frac{1}{\sqrt{x}} \, dx = 1 + \int_1^\infty \frac{1}{x^2} \, dx
\]
and confirm, using the values you already know for the integrals above, that the equation does in fact hold.
Problem E: The Ultraviolet Catastrophe

a. Explain why \( 1 + x \leq e^x \) for all values of \( x \).

*Hint:* \( y = 1 + x \) is tangent to \( y = e^x \) at \( x = 0 \), but this is not enough. What is true about the second derivative of \( e^x \)? (Graphing these functions should help you see what is going on.)

b. Use part (a) to show that

\[
e^{1/x} - 1 \geq \frac{1}{x}
\]

for all \( x \neq 0 \).

c. Show that the integral

\[
\int_1^\infty \frac{dx}{x^5(e^{1/x} - 1)}
\]

converges (do not attempt to find its value).

The integral in part (c) figures in the resolution of a problem that frustrated physicists in the early 20th century called the Ultraviolet Catastrophe. In brief, classical models of physics predicted that certain objects called blackbodies would emit infinite amounts of energy as radiation, contradicting both physical intuition and experimental observations. Max Planck settled the issue by appealing to a quantum model, which would instead give the emitted energy in terms of the convergent improper integral above—his calculation fit the data perfectly. In solving the Ultraviolet Catastrophe, Planck provided yet more evidence for the necessity and predictive utility of quantum mechanics.

Problem F: Asymptotic Relations

In problems (a–h) below, fill in the blank with the correct relation: \( <, \approx, \text{ or } > \). If you are confident in your answer, you do not need to show your work (i.e., by taking limits of \( \lim_{x \to \infty} \frac{|f(x)|}{g(x)} \)). However, incorrect answers with some work attached might receive partial credit.

a. \( x \ldots 2x^3 + 1 \)

b. \( 2^{-x} \ldots e^{-x} \)

c. \( x^2 \ln x \ldots 10^{256} \cdot x^2 \)

d. \( \ln x \ldots \sqrt{x} \)

e. \( x \ln x \ldots (\ln x)^2 \)

f. \( x \ldots \sqrt{4x^2 + 1} \)

g. \( \ln(x^2) \ldots \ln x \)

h. \( \frac{1}{x^2} \ldots \frac{x + 1}{x^5} \)