\[ a : \ f(x) = 1 \quad \text{and} \quad g(x) = x. \]

We have \( f(x) < g(x) \), but
\[
\int_1^\infty \frac{f(x)}{g(x)} \, dx = \int_1^\infty \frac{1}{x} \, dx
\]

diverges by \( p \)-test.

\[ b : \text{We have} \quad \frac{x}{e^x} < \frac{1}{x^2}, \quad \text{because:} \]
\[
\lim_{x \to \infty} \frac{x/e^x}{1/x^2} = \lim_{x \to \infty} \frac{x^3}{e^x}
\]

\[ = 0, \quad \text{because we know that} \ x^3 < e^x. \]

(cor by L'Hôpital's Rule)

Thus \( \int_1^\infty \frac{x}{e^x} \, dx \) converges, because \( \int_1^\infty \frac{1}{x^2} \, dx \) does.

\[ \text{Alternate argument:} \quad \frac{1}{e^x} < \frac{1}{x^2} \Rightarrow \frac{x}{e^x} < \frac{x}{x^3} = \frac{1}{x^2}. \]

\[ \int_0^1 \ln x \, dx = \lim_{a \to 0^+} \int_a^1 \ln x \, dx
\]

\[ = \lim_{a \to 0^+} \left[ x \ln x - x \right]_a
\]

\[ = \lim_{a \to 0^+} \left[ \ln(1) - 1 - a \ln a + a \right]
\]

\[ = 0 - 1 - 0 + 0 = -1 \]

\[ \text{\( \star \):} \lim_{a \to 0^+} \left[ a \ln a \right] = "0 \cdot (-\infty)" \quad \text{indeterminate}
\]

\[ = \lim_{a \to 0^+} \left[ \frac{\ln a}{1/a} \right] = \frac{-\infty}{+\infty}
\]

Can now apply L'Hôpital's Rule:

\[ = \lim_{a \to 0^+} \left[ \frac{1/a}{-1/a^2} \right] = \lim_{a \to 0^+} \left[ \frac{a^2}{a} \right]
\]

\[ = \lim_{a \to 0^+} \left[ -a \right] = 0. \]

\[ \text{In graph,} \quad A \text{ and} \ B \text{ are congruent \( (B \Leftarrow A) \) reflected over} \ y=x \text{ b/c} \ln x \text{ and} \ e^x \text{ are inverse functions.} \]
2) b: Since they are congruent, they have the same area (but $\int_0^1 \ln x \, dx$ is negative, the region is below the x-axis).

Thus $\int_0^1 \ln x \, dx = -\int_0^\infty e^x \, dx$.

Now, we evaluate

\[ \int_{-\infty}^0 e^x \, dx = \lim_{a \to -\infty} \int_a^0 e^x \, dx \]

\[ = \lim_{a \to -\infty} \left[ e^x \right]_a^0 \]

\[ = 1 - e^{-\infty} = 1 - 0 = 1 \]

Finally, using the formulas for $\int_0^1 \frac{1}{x^p} \, dx$ and $\int_1^\infty \frac{1}{x^p} \, dx$, we have

\[ \frac{1}{1 - \frac{1}{p}} = \frac{p}{p-1} \text{ on the left, and} \]

\[ 1 + \frac{1}{p-1} = \frac{(p-1) + 1}{p-1} = \frac{p}{p-1} \text{ on the right}. \]

d: We know $\int_0^1 \frac{1}{x^q} \, dx$ converges when $0 < q < 1$. Shifting to the right by $a$,
we see that $\int_a^{a+1} \frac{1}{(x-a)^q} \, dx$ converges too.

\[ \int_b^{a+1} \frac{1}{(x-a)^q} \, dx \text{ converges.} \]

Finally, changing $a + 1$ to $b > a$ does not "pick up" any asymptotes, so

\[ \int_a^{a+1} \frac{1}{(x-a)^q} \, dx \text{ converges too.} \]
3. If \( f(x) \times g(x) \), then

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = c, \quad \text{where } c \neq 0, \text{ finite}
\]

then

\[
\lim_{x \to \infty} \frac{1/f(x)}{1/g(x)} = \lim_{x \to \infty} \frac{g(x)}{f(x)} = \frac{1}{c}
\]

and \( \frac{1}{c} \neq 0 \) and finite.

b. True. We have

\[
\lim_{x \to \infty} \frac{f(x) + g(x)}{g(x)} = \lim_{x \to \infty} \left[ \frac{f(x)}{g(x)} + 1 \right] = 0 + 1 = 1.
\]

4. The function \( e^x - (x+1) \) must occur at a point where \( f'(x) = 0 \).

We have \( f'(x) = e^x - 1 \), so the only possibility is at \( x = 0 \).

Furthermore

- It is a minimum, \( b/c \)

\[
f''(0) = e^0 = 1 \text{ is positive} \]

(2nd derivative test)

The value of \( e^0 - (0+1) = 0 \)

Therefore, since 0 is the minimum value of \( e^x - (x+1) \), \( e^x - (x+1) \geq 0 \)

\[
e^x \geq x + 1
\]

b. If \( t \neq 0 \), then \( \frac{t}{x} \) is a real #, so we can plug it in for \( x \) in \( \infty \).

c. \[
L \]
We have
\[ e^{1/t} - 1 \geq \frac{1}{t} \geq 0, \quad \text{for } t \geq 1 \]

\[ \Downarrow \text{taking reciprocals} \]
\[ \frac{1}{t} \leq e^{1/t} - 1 \leq t \quad \text{for } t \geq 1 \]

\[ \Downarrow \text{multiply by } \frac{1}{t^5}, \text{ which is positive} \]
\[ \frac{1}{t^5(e^{1/t} - 1)} \leq \frac{t}{t^5} = \frac{1}{t^4} \]

So
\[ 0 \leq t^4(e^{1/t} - 1) \leq \frac{1}{t^4} \]

\[ \Downarrow \int_1^\infty \]
\[ \int_1^\infty \frac{dt}{t^5(e^{1/t} - 1)} \leq \int_1^\infty \frac{1}{t^4} dt \]

\[ \uparrow \text{converges, so} \]
\[ \int_1^\infty \frac{dt}{t^5(e^{1/t} - 1)} \]

Thus:
\[ \int_0^1 e^{-t^2} dt = \int_0^{\sqrt{n}} e^{-u^2} \frac{du}{\sqrt{n}} \]

\[ = \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-u^2} du \]
4. 7.6.16 in 6th edition, 7.6.18 in 7th edition:

\[
\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}} \, dx = \lim_{b \to \pi/2^-} \int_{\pi/4}^{b} \frac{\sin x}{\sqrt{\cos x}} \, dx \\
= \lim_{b \to \pi/2^-} \left[ \frac{\sin x}{-(\cos x)^{1/2}} \right]_{\pi/4}^{b} \\
= \lim_{b \to \pi/2^-} \left[ -2(\cos x)^{1/2} \right]_{\pi/4}^{b} \\
= \lim_{b \to \pi/2^-} \left[ -2(\cos b)^{1/2} + 2(\cos \pi/4)^{1/2} \right] \\
= 2 \left( \frac{\sqrt{2}}{2} \right)^{1/2} = 2^{3/4}.
\]

5. 7.6.24 in 6th edition, 7.6.28 in 7th edition:

With the substitution \( w = \ln x \), \( dw = \frac{1}{x} \, dx \),

\[
\int_{0}^{1} \frac{\ln x}{x} \, dx = \int w \, dw = \frac{1}{2} w^2 + C = \frac{1}{2} (\ln x)^2 + C
\]

so

\[
\int_{0}^{1} \ln x \, dx = \lim_{a \to 0^+} \int_{a}^{1} \frac{\ln x}{x} \, dx = \lim_{a \to 0^+} \frac{1}{2} |\ln(x)|^2 \bigg|_{a}^{1} = \lim_{a \to 0^+} \frac{-1}{2} |\ln(a)|^2.
\]

As \( a \to 0^+ \), \( \ln a \to -\infty \), so the integral diverges.

6. 7.6.26 in 6th edition, 7.6.30 in 7th edition:

Using the substitution \( w = -x^{1/2} \), \( -2dw = x^{-1/2} \, dx \),

\[
\int e^{-\frac{1}{2}x} \, dx = -2 \int e^w \, dw = -2e^{-\frac{1}{2}x} + C.
\]

So

\[
\int_{0}^{\pi} \frac{1}{\sqrt{x}} e^{-\sqrt{x}} \, dx = \lim_{b \to 0^+} \int_{b}^{\pi} \frac{1}{\sqrt{x}} e^{-\sqrt{x}} \, dx \\
= \lim_{b \to 0^+} -2e^{-\sqrt{x}} \bigg|_{b}^{\pi} \\
= 2 - 2e^{-\sqrt{\pi}}.
\]

7. 7.6.28 in 6th edition, 7.6.32 in 7th edition:

\[
\int_{0}^{2} \frac{1}{\sqrt{4-x^2}} \, dx = \lim_{b \to 2^-} \int_{0}^{b} \frac{1}{\sqrt{4-x^2}} \, dx \\
= \lim_{b \to 2^-} \arcsin \left( \frac{x}{2} \right) \bigg|_{0}^{b} \\
= \lim_{b \to 2^-} \arcsin \frac{b}{2} = \arcsin 1 = \frac{\pi}{2}.
\]

8. 7.7.14 in 6th edition, 7.7.18 in 7th edition:

The integral converges.

\[
\int_{0}^{1} \frac{1}{x^{19/20}} \, dx = \lim_{a \to 0^+} \int_{a}^{1} \frac{1}{x^{19/20}} \, dx = \lim_{a \to 0} 20x^{1/20} \bigg|_{a}^{1} = \lim_{a \to 0} 20 \left( 1 - a^{1/20} \right) = 20.
\]
9. **7.7.20 in 6th edition, 7.7.24 in 7th edition:**

This integral is improper at $\theta = 0$. For $0 \leq \theta \leq 1$, we have $\frac{1}{\sqrt{\theta^2 + \theta}} \leq \frac{1}{\sqrt{\theta}}$ and since $\int_0^1 \frac{1}{\sqrt{\theta}} d\theta$ converges,

$$\int_0^1 \frac{d\theta}{\sqrt{\theta^2 + \theta}}$$ converges.

The inequality in the above solution can be obtained using the problem bystander approach: $\frac{1}{\sqrt{\theta^2 + \theta}} = \frac{1}{\sqrt{\theta}} \cdot \frac{1}{\sqrt{\theta^2 + 1}}$ and the bystander $\frac{1}{\sqrt{\theta^2 + 1}}$ has max value $= 1$ for $0 \leq \theta \leq 1$.

10. **7.7.22 in 6th edition, 7.7.26 in 7th edition:**

This integral is convergent because, for $\phi \geq 1$,

$$\frac{2 + \cos \phi}{\phi^2} \leq \frac{3}{\phi^2},$$

and $\int_1^\infty \frac{3}{\phi^2} d\phi = 3 \int_1^\infty \frac{1}{\phi^2} d\phi$ converges.

The inequality above and in the next solution can both be obtained using $-1 \leq \sin \phi, \cos \phi \leq 1$.

11. **7.7.24 in 6th edition, 7.7.28 in 7th edition:**

Since $\frac{1}{\phi^2} \leq \frac{2 - \sin \phi}{\phi^2}$ for $0 < \phi \leq \pi$, and since $\int_0^\pi \frac{1}{\phi^2} d\phi$ diverges, $\int_0^\pi \frac{2 - \sin \phi}{\phi^2} d\phi$ must diverge.