1. (7.6.14) We note that the integral has $\pm \infty$ for both bounds and that the integrand is nonsingular on all of $\mathbb{R}$. Thus, we can split the domain of integration at 0 to get two improper integrals of Type I:

$$
\int_{-\infty}^{\infty} \frac{dz}{z^2 + 25} = \int_{-\infty}^{0} \frac{dz}{z^2 + 25} + \int_{0}^{\infty} \frac{dz}{z^2 + 25}
$$

$$
= \lim_{A \to -\infty} \int_{A}^{0} \frac{dz}{z^2 + 25} + \lim_{B \to \infty} \int_{0}^{B} \frac{dz}{z^2 + 25}
$$

$$
= \frac{1}{5} \arctan \left( \frac{z}{5} \right) \bigg|_{A}^{0} + \frac{1}{5} \arctan \left( \frac{z}{5} \right) \bigg|_{0}^{B}
$$

$$
= \frac{1}{5} \left( \lim_{A \to -\infty} \arctan(0) - \arctan \left( \frac{A}{5} \right) \right) + \frac{1}{5} \left( \lim_{B \to \infty} \arctan \left( \frac{B}{5} \right) - \arctan(0) \right)
$$

$$
= -\frac{1}{5} \lim_{A \to -\infty} \arctan \left( \frac{A}{5} \right) + \lim_{B \to \infty} \arctan \left( \frac{B}{5} \right)
$$

$$
= \frac{1}{5} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{\pi}{5}
$$

2. (7.6.16) The integrand here becomes singular when $\cos x = 0$. The only value where this occurs within the domain of integration is at $x = \frac{\pi}{2}$. Thus we have an improper integral of Type II and

$$
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{\cos x}} \, dx = \lim_{A \to \frac{\pi}{2}} \int_{A}^{\frac{\pi}{4}} \frac{\sin x}{\sqrt{\cos x}} \, dx.
$$

We can compute this integral with a $u$-substitution, where $u = \cos x$ and $du = -\sin x \, dx$ (so that $-du = \sin x \, dx$). That is,

$$
\lim_{A \to \frac{\pi}{2}} \int_{A}^{\frac{\pi}{4}} \frac{\sin x}{\sqrt{\cos x}} \, dx = \lim_{A \to \frac{\pi}{2}} \int_{\frac{\cos A}{\sqrt{2}}}^{\frac{\cos \frac{\pi}{4}}{\sqrt{2}}} \frac{-du}{u} = \lim_{A \to \frac{\pi}{2}} -2u^{\frac{1}{2}} \bigg|_{\frac{\cos A}{\sqrt{2}}}^{\frac{\cos \frac{\pi}{4}}{\sqrt{2}}} = -2 \left( \lim_{A \to \frac{\pi}{2}} (\cos B) \right)^{\frac{1}{2}} - \left( \frac{\sqrt{2}}{2} \right)^{\frac{1}{2}}
$$

$$
= -2 \left( 0 - \left( \frac{\sqrt{2}}{2} \right)^{\frac{1}{2}} \right) = 2^3 2^3 = 2^3
$$

3. (7.6.18) We first note that

$$
\int_{0}^{1} \frac{x^4 + 1}{x} \, dx = \int_{0}^{1} \frac{1}{x} \, dx + \int_{0}^{1} x^3 \, dx + \int_{0}^{1} \frac{1}{x} \, dx
$$

This first integral is not improper and can just be evaluated directly. In the second integral, the integrand becomes singular when $x = 0$ and so is improper of Type 2. Thus,

$$
\int_{0}^{1} x^3 \, dx + \int_{0}^{1} \frac{1}{x} \, dx = \frac{1}{4} x^4 \bigg|_{0}^{1} + \lim_{B \to 0} \int_{B}^{1} \frac{1}{x} \, dx = \frac{1}{4} + \lim_{B \to 0} \ln x \bigg|_{B}^{1} = \frac{1}{4} + \ln 1 - \lim_{B \to 0} \ln B.
$$

This limit diverges, so the original integral diverges.

4. (7.6.28) The integrand here becomes singular only when $x = 2$. As the bounds of integration are finite, this is an improper integral of Type II. Thus,

$$
\int_{0}^{2} \frac{1}{\sqrt{4 - x^2}} \, dx = \lim_{B \to 2} \int_{0}^{B} \frac{1}{\sqrt{4 - x^2}} \, dx = \lim_{B \to 2} \arcsin \left( \frac{x}{2} \right) \bigg|_{0}^{B} = \lim_{B \to 2} \arcsin \left( \frac{B}{2} \right) - \arcsin 0 = \arcsin 1 = \frac{\pi}{2}.
$$
5. (7.6.32) The integrand here becomes singular only when \( x = 3 \). As the bounds of integration are finite, this is an improper integral of Type II. Thus,

\[
\int_0^3 \frac{y}{\sqrt{9 - y^2}} \, dy = \lim_{B \to 3} \int_0^B \frac{y}{\sqrt{9 - y^2}} \, dy
\]

We can evaluate this integral with a \( u \)-substitution with \( u = 9 - y^2 \) and \( du = -2y \, dy \) (so that \( y \, dy = \frac{-1}{2} \, du \)). With this substitution,

\[
\lim_{B \to 3} \int_0^B \frac{y}{\sqrt{9 - y^2}} \, dy = \lim_{B \to 3} \int_9^{B - B^2} u^{-\frac{1}{2}} \frac{-du}{2} = -\lim_{B \to 3} u^{\frac{1}{2}} \frac{9 - B^2}{9} = -\lim_{B \to 3} (\sqrt{9 - B^2} - 3) = 3
\]

and the integral converges.

6. (7.6.50) This statement is FALSE. For a counterexample, note that \( \int_1^\infty \frac{1}{x} \, dx \) diverges even though \( \lim_{x \to \infty} \frac{1}{x} = 0 \).

7. (7.7.12) Note that

\[
\lim_{x \to 1} \frac{1}{x^3 + 1} = \lim_{x \to 1} \frac{x^3}{x^3 + 1} = \frac{1}{1 + \frac{1}{x^3}} = 1.
\]

Thus, \( \frac{1}{x^3 + 1} \sim \frac{1}{x^3} \). The Limit Comparison Test says that because the integral of the latter function converges (by the \( p \)-test), the integral of the former converges as well.

8. (7.7.20) Note that

\[
f(x) = \frac{1}{\sqrt{\theta^3 + \theta}} \leq \frac{1}{\sqrt{\theta}} = g(x)
\]

on the domain of integration. At the inequality step, we have made a denominator smaller. But

\[
\int_0^1 \theta^{-\frac{1}{2}} \, d\theta = \lim_{B \to 0^+} \int_B^1 \theta^{-\frac{1}{2}} \, d\theta = 2 \lim_{B \to 0^+} \theta^{\frac{1}{2}} \bigg|_B^1 = 2(\lim_{B \to 0^+} 1 - B^{\frac{1}{2}}) = 2,
\]

so this integral converges. As the integral of \( g(x) \) converges and \( f(x) < g(x) \), the integral of \( g(x) \) converges as well.

9. (7.7.22) First note that \( |\cos \phi| \leq 1 \), so \( 1 \leq 2 - \cos \phi \leq 3 \). Hence,

\[
f(x) = \frac{2 - \cos \phi}{\phi^2} \leq \frac{3}{\phi^2} = g(x).
\]

We know that the integral of \( g(x) \) converges (by the \( p \)-test), so the comparison test implies that the integral of \( f(x) \) converges as well.

10. (7.7.34) This is FALSE. For a counterexample, note that \( 0 \leq \frac{1}{x^2} \leq \frac{1}{x} \) but \( \int_1^\infty \frac{1}{x} \, dx \) diverges while \( \int_1^\infty \frac{1}{x^2} \, dx \) converges.

11. (Problem D)

(a) We use the given formula to compute that

\[
\int_0^1 \ln x \, dx = \lim_{B \to 0^+} \int_B^1 \ln x \, dx = \lim_{B \to 0^+} x \ln x - x \bigg|_B^1 = \lim_{B \to 0^+} -1 - B \ln B + B
\]

\[
= \lim_{B \to 0^+} -1 - \frac{\ln B - 1}{B} = \lim_{B \to 0^+} -1 - \frac{1}{B} = \lim_{B \to 0^+} -1 - B = -1.
\]
(b) The functions $f(x) = \ln x$ and $g(x) = e^x$ satisfy the relation that $f(g(x)) = g(f(x)) = x$. That is to say, $f$ and $g$ are inverses under composition of functions. Given a function $f$, the graph of its inverse $g$ is given by reflection across the line $y = x$. Thus, the two integrals here represent areas related by a reflection in the plane and so have the same value. For the computation, we have that

$$-\int_{-\infty}^{0} e^x \, dx = -\lim_{B \to -\infty} \int_{B}^{0} e^x \, dx = -\lim_{B \to -\infty} e^x \bigg|_{B}^{0} = -(\lim_{B \to -\infty} 1 - e^B) = -1.$$

(c) As in the previous part, the two integrands here are inverse functions and the areas in question are thus related by a reflection in the plane, with the exception of the square with vertices $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ which appears in the left hand integral but not the right. This square has area 1, which accounts for the additional term on the right-hand side. We can verify this as

$$1 + \int_{1}^{\infty} \frac{1}{x^2} \, dx = 1 + \lim_{B \to \infty} \int_{1}^{B} x^{-2} \, dx = 1 - \lim_{B \to \infty} x^{-1} \bigg|_{1}^{B} = 1 - (\lim_{B \to \infty} B^{-1} - 1) = 1 + 1 = 2.$$ 

12. (Problem E)

(a) If we set $y(x) = 1 + x$ and $f(x) = e^x$, then it is easy to see that $y'(x) = 1$, $y''(x) = 0$ and $f'(x) = f''(x) = e^x$ for all $x$. Thus, we see that $y(x)$ is tangent to $f(x)$ at $x = 0$. Further, the second derivative of $f$ is positive everywhere so that the graph of $f$ is concave up. This implies that the graph of $f(x)$ must always lie above the graph of $y(x)$. This exactly says that $y(x) \geq f(x)$.

(b) Similar to the previous part, we can do the substitution $x \rightarrow \frac{1}{x^2}$ in the inequality of Problem E (a) to see that $1 + x^2 \geq e^{1/x^2}$, or equivalently that $e^{1/x^2} \geq 1 + x^2$.

(c) The previous inequality can be rearranged to read

$$\frac{1}{e^{1/x} - 1} \leq x$$

so that

$$\frac{1}{x^5(e^{1/x} - 1)} \leq \frac{x}{x^5} = \frac{1}{x^4}.$$ 

But we know that $\int_{1}^{\infty} \frac{1}{x^4} \, dx$ converges (by the $p$-test), so the integral of our original function converges by the comparison test.
Supplemental problem F

a. \( x < 2x^3 + 1 \).
   Explanation: when both sides are polynomial, the one with higher degree dominates.

b. \( 2^{-x} \gg e^{-x} \).
   Explanation: this is the same as saying \((1/2)^x \gg (1/e)^x\), which is true because when both sides are of the form \(a^x\), the one with bigger base dominates.

c. \( x^2 \ln x > 10^{256}, x^2 \).
   Explanation: divide both sides by \(x^2\), and see that \(\ln x > 10^{256}\). Note that you can always divide both sides by the same function, but you can NOT subtract both side by the same function.

d. \( \ln x \ll \sqrt{x} \).
   Explanation: logarithms grow slower than any positive power of \(x\).

e. \( x \ln x \gg (\ln x)^2 \).
   Explanation: divide both sides by \(\ln x\), and see that \(x \gg \ln x\).

f. \( x \gg \sqrt{4x^2 + 1} \).
   Explanation: note that \(\frac{x}{\sqrt{4x^2 + 1}} = \frac{1}{\sqrt{4 + \frac{1}{x^2}}}\), so \(\lim_{x \to \infty} \frac{x}{\sqrt{4x^2 + 1}} = \frac{1}{\sqrt{4}} = \frac{1}{2}\).

g. \( \ln(x^2) \approx \ln x \).
   Explanation: \(\ln(x^2) = 2\ln x\).
h. \( \frac{1}{x^2} \geq \frac{x+1}{x^2} \).

Explanation: \( \lim_{x \to \infty} \frac{1}{x^2} / \frac{x+1}{x^2} = \lim_{x \to \infty} \frac{x^3}{x+1} = \infty \). Here the last equality holds because polynomials of higher degrees dominate ones of lower degrees.