Math 21, Winter 2018 — Schaeffer/Wieczorek
Stanford University

Homework 3, due Friday, February 8th, at 4:00 PM on Gradescope

Information and instructions

Most of the material in this assignment was covered in the lectures of 1/25 and 2/01 (the Fridays before and after Midterm 1). This corresponds to Sections 9.1 and 9.2 of the textbook. The required reading for this assignment is those two sections.

A few problems concern material in Section 9.3, which will be covered in the lecture of 2/04 (Monday). These problems are 4, 5, and 10.

Conceptual questions

1. Decide if the following statements are true or false and explain your answers:
   a. You can tell if a sequence converges by looking at the first million terms.
   b. If the terms of a convergent sequence are all > 0, then the limit of the sequence will also be > 0.
   c. If a sequence of positive terms is not bounded, then the sequence must contain a term that is \( \geq 100^{100} \).
   d. If a sequence of positive terms is not bounded, then the sequence must contain infinitely many terms that are \( \geq 100^{100} \).
   e. If a convergent sequence \( \{a_n\} \) satisfies \( 0 \leq a_n \leq 1 \) for every index \( n \), then we will have \( 0 \leq \lim_{n \to \infty} (a_n) \leq 1 \) as well.
   f. A monotone sequence cannot have both positive and negative terms.

2. Explain what is wrong with the following statements:
   a. The geometric sequence \( 4, 1, \frac{1}{4}, \frac{1}{16}, \ldots \) converges to \( \frac{4}{1-1/4} = \frac{16}{3} \).
   b. The geometric series \( 1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots \) converges to \( \frac{1}{1-(3/2)} = \frac{2}{5} \).
   c. The following geometric series is convergent: \( 0.00001 + 0.0001 + 0.001 + \cdots \).

3. Suppose that \( \lim_{k \to \infty} (a_k) = 1 \). Explain why the series \( \sum_{k=1}^{\infty} a_k \) diverges.

4. Draw a graph that clearly illustrates the inequality \( \sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{1}{x} \, dx \) for \( n = 5 \).

5. Draw a graph that clearly illustrates the inequality \( \sum_{k=1}^{n} \frac{1}{k^2} \leq 1 + \int_{1}^{n} \frac{1}{x^2} \, dx \) for \( n = 5 \).
Routine problems

6. For each of the sequences below, (i) find a closed formula for the terms (no $\sum$ allowed in your formula!), (ii) decide whether it is convergent (and find its limit if so), (iii) decide whether it is bounded, (iv) decide whether it is monotone, and (v) decide whether it is geometric.

a. $1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \ldots$

b. $\frac{1}{2 \cdot 1'} \frac{1}{3 \cdot 2'} \frac{1}{4 \cdot 3'} \frac{1}{5 \cdot 4'} \frac{1}{6 \cdot 5'} \ldots$

c. $1, -1, 1, -1, 1, -1, 1, -1, \ldots$

d. $\frac{5}{2}, \frac{5}{\sqrt{2}}, 5, \ldots$

e. $1, 1 - \frac{1}{3}, 1 - \frac{1}{3} + \frac{1}{9}, 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27}, \ldots$

f. $2, \sqrt{2}, \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \ldots$

7. The sequence $\left\{ \frac{4n + (-1)^n \cdot 5}{5n + 1} \right\}_{n=0}^{\infty}$ converges.

a. Explain why you cannot apply L'Hôpital’s rule to evaluate the limit of this sequence, even though the numerator and denominator both tend to $+\infty$.

Hint: It might be helpful to find the statement of L’HR in the textbook and check the “fine print.”

b. Demonstrate how to find the limit of the sequence using another method.

8. Each of the sums/series below are geometric. For each, (i) determine the value of $r$ (the ratio between terms), (ii) write the sum in $\sum$ notation, and (iii) evaluate the sum using the geometric sum formula, and write your answer as a whole number or a fraction in lowest terms.

a. $1 - 2 + 4 - 8 + 16 - 32 + \cdots + 1024$

b. $60 + 20 + \frac{20}{3} + \frac{20}{9} + \cdots + \frac{20}{729}$

c. $0.1 + 0.01 + 0.001 + \cdots$

d. $0.01 + 0.0001 + 0.000001 + \cdots$

e. $1 + \frac{5}{3} + \frac{25}{9} + \cdots$

f. $2 \cdot 3 + \frac{2}{3} + \frac{2}{3^2} + \cdots$

9. Write the repeating decimal $0.\overline{73} = 0.737373\ldots$ as a fraction in lowest terms (you can check your work using a calculator if you like!). Hint: Use Problem 8.

10. Complete the following exercises from Section 9.3 of the textbook (either edition):

a. Exercise 4

b. Exercise 6

c. Exercise 8
In-depth problems

11. The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. That is, each term is the sum of the previous two terms.

   a. Write down the first 13 terms of the Fibonacci sequence.

   b. Though the Fibonacci sequence is not literally geometric, the limit

   $$\varphi = \lim_{n \to \infty} \left( \frac{F_{n+1}}{F_n} \right)$$

   exists and is positive. Using the fact that this limit exists, find an exact value for $\varphi$.

   **Hint 1:** Remember that $F_{n+1}$ is the sum of the two previous terms.

   **Hint 2:** If $\frac{F_{n+1}}{F_n} \to \varphi > 0$ as $n \to \infty$, then what is the limit of $\frac{F_{n+1}}{F_n}$ as $n \to \infty$?

   **Hint 3:** Perhaps unexpectedly, you’re going to need the quadratic formula! The quadratic formula sometimes gives you two answers—in this case only one of them is correct and you should justify your choice.

   c. By (b), $F_{n+1} \approx \varphi \cdot F_n$ for large indices $n$, so $F_n \asymp \varphi^n$ as $n \to \infty$. More precisely, we have the formula

   $$F_n = \left[ \frac{\varphi^n}{\sqrt{5}} \right]$$

   where $[y]$ means “$y$ rounded to the nearest integer.” Use this (and a calculator with an exponentiation function) to find $F_{20}$, the 20th Fibonacci number.

   d. Though he was not the first to study this sequence, it is named after the 12th century Italian mathematician Leonardo “Fibonacci” of Pisa. He discovered the sequence while imagining an idealized population growth model for rabbits. (Look it up, if you’re curious.)

   Suppose that PLANET R is home to robots and rabbits. If the number of robots after $n$ weeks is $2^n$ and the number of rabbits after $n$ weeks is $F_n$, then who will eventually dominate the planet? That is, one of

   $$\lim_{n \to \infty} \left( \frac{\text{robots}}{\text{robots} + \text{rabbits}} \right) \quad \text{or} \quad \lim_{n \to \infty} \left( \frac{\text{rabbits}}{\text{robots} + \text{rabbits}} \right)$$

   will be equal to 1. Which, and why? Alternatively, you can show (by taking a limit) that either robots ≺ rabbits or robots ≻ rabbits.

12. This problem picks up where another in-depth problem in HW2 left off. Our ultimate goal is to establish the formula $\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$ (Gauss’ integral) which is of vital importance in statistics and probability.
At the end of HW2, we concluded with the inequality
\[ \sqrt{n} \int_0^1 (1 - x^2)^n \, dx \leq \int_0^{\pi/2} (\cos \theta)^{2n+1} \, d\theta \leq \sqrt{n} \int_0^1 \frac{dx}{(x^2 + 1)^n} \]  

(*)

Our intent is to take the limit of the above inequality as \( n \to \infty \)—the middle expression become \( \int_0^\infty e^{-x^2} \, dx \), which is Gauss’ integral.

The next step in our analysis is to consider the integrals on the left and right of (*) By employing the integration technique of trigonometric substitution (which is certainly not required knowledge for Math 21), we have

- \( \int_0^1 (1 - x^2)^n \, dx = \int_0^{\pi/2} (\cos \theta)^{2n+1} \, d\theta \) and
- \( \int_0^1 \frac{dx}{(x^2 + 1)^n} = \int_0^{\pi/4} (\cos \theta)^{2n-2} \, d\theta \leq \int_0^{\pi/2} (\cos \theta)^{2n-2} \, d\theta. \)

If you’re curious how the trig substitutions work: in the first integral we made the substitution \( x = \sin \theta \), so \( 1 - x^2 = (\cos \theta)^2 \) and \( dx = \cos \theta \, d\theta \); in the second we use the substitution \( x = \tan \theta \) so \( \frac{1}{1+x^2} = (\cos \theta)^2 \) and \( dx = \frac{d\theta}{(\cos \theta)^2}. \)

Combining with the inequality (*) above, we obtain
\[ \sqrt{n} \cdot I_{2n+1} \leq \int_0^{\pi/2} e^{-x^2} \, dx \leq \sqrt{n} \cdot I_{2n-2} \]

where \( I_k = \int_0^{\pi/2} (\cos \theta)^k \, d\theta. \) The problems below concern the sequences \( \{I_k\}_{k=0}^{\infty} \) and \( \{I_k \cdot I_{k+1}\}_{k=0}^{\infty}. \)

a. Show that \( I_0 = \frac{\pi}{2} \) and that \( I_1 = 1. \)

b. Using the book’s integration table, show that the sequence \( \{I_k\} \) satisfies the “two steps back” recurrence \( I_k = \frac{k-1}{k} I_{k-2} \) (so long as \( k \geq 2 \)).

Note: Some printed copies of the textbook have an incomplete integration table—the complete version is on the course website.

c. Use (a.) and (b.) to compute the values of \( I_0, I_1, I_2, \ldots, I_7. \) You should express your answers either as nice fractions (for the odd-indexed terms), or as nice fraction multiples of \( \frac{\pi}{2} \) (for the even-indexed terms).

d. \( \{I_k\} \) is in fact a monotone sequence. Based on the values you found in (c.), is \( \{I_k\} \) increasing or decreasing?

e. Using the values you found in (c.), find a nice formula for \( I_k \cdot I_{k+1} (I_k \text{ *times* } I_{k+1}) \) in terms of \( k. \) Hint: It’s easier to see what the formula will be if you write each \( I_k \cdot I_{k+1} \) as a nice fraction multiple of \( \frac{\pi}{2}. \)

The exciting conclusion in Homework 4.