Solutions to homework 3

Book problems - section 9.1

Exercise 22  Note that because \((-1)^n\) is bounded, \(\lim_{n \to \infty} \frac{(-1)^n}{n} = 0\). Thus,
\[
\lim_{n \to \infty} \frac{2n + (-1)^n5}{4n - (-1)^n3} = \lim_{n \to \infty} \frac{2 + \frac{(-1)^n}{n} \cdot 5}{4 - \frac{(-1)^n}{n} \cdot 3} = \frac{2 + 0}{4 - 0} = \frac{1}{2}.
\]

Exercise 26

(a) \(\lim_{n \to \infty} n(n + 1) - 1 = +\infty\), so this matches with (II).

(b) \(\lim_{n \to \infty} \frac{1}{n+1} = 0\) and \(\frac{1}{n+1} > 0\) for every \(n\), so this matches with (III).

(c) \(\lim_{n \to \infty} 1 - n^2 = -\infty\), so this matches with (I).

(d) Since \(\lim \frac{1}{n} = 0\) and \(\cos(x)\) is continuous, \(\lim_{n \to \infty} \cos(\frac{1}{n}) = \cos(0) = 1\), so this matches with (IV).

(e) Since \(\sin n\) is bounded, \(\lim_{n \to \infty} \frac{\sin n}{n} = 0\). Moreover, as \(n\) gets bigger, \(\sin n\) goes through both positive and negative values, hence so does \(\frac{\sin n}{n}\). So this matches with (V).

Book problems - section 9.2

Exercise 24  The formula for the sequence is \(a_n = 10 \cdot (-\frac{2}{3})^n\). If you wonder where this comes from, the ratio \(-\frac{2}{3}\) can be found by taking the ratio of any two consecutive terms: \(\frac{\frac{540}{810}}{-\frac{360}{540}} = \cdots = -\frac{2}{3}\). The base \(-810\) is just the value of the first term. So,
\[
\sum_{n=0}^{\infty} -810 \left(-\frac{2}{3}\right)^n = -810 \cdot \frac{1}{1 - \left(-\frac{2}{3}\right)} = -810 \cdot \frac{3}{5} = -486.
\]

Exercise 26  The formula for the sequence is \(a_n = \left(\frac{z}{2}\right)^n\). The partial sum is
\[
\sum_{n=0}^{M} \left(\frac{z}{2}\right)^n = \frac{1 - \left(\frac{z}{2}\right)^{M+1}}{1 - \frac{z}{2}}.
\]
Thus, if \(-2 < z < 2\) then \(|\frac{z}{2}| < 1\), so the series converges to \(\frac{1}{1-\frac{z}{2}}\). Otherwise, the series diverges.

Exercise 46

(a) \(h_n = 10 \cdot \left(\frac{3}{4}\right)^n\).
(b) Note that the ball has to travel downward 10 feet before it hits the ground for the first time, and that the distance traveled between the \( k \)th and \((k + 1)\)th time is twice the height \( h_k \), since the ball has to move upward and then forward. Thus, the total distance the ball has traveled when it hits the floor for the \( n \)th time is:

- For \( n = 1 \) : 10 feet.
- For \( n = 2 \) : \( 10 + 2 \cdot h_1 = 10 + 2 \cdot 7.5 = 25 \) feet.
- For \( n = 3 \) : \( 10 + 2 \cdot h_1 + 2h_2 = 10 + 2 \cdot 7.5 + 2 \cdot 5.625 = 36.25 \) feet.
- For general \( n \geq 2 \):

\[
10 + 2h_1 + 2h_2 + \cdots + 2h_{n-1} = 10 + \sum_{k=1}^{n-1} 2 \cdot 10 \cdot \left( \frac{3}{4} \right)^k = 10 + 20 \sum_{k=1}^{n-1} \left( \frac{3}{4} \right)^k
\]

\[
= 10 + 20 \cdot \frac{3^{n-2}}{4} \left( \frac{3}{4} \right)^k = 10 + 20 \cdot \frac{3}{4} \cdot \frac{1 - \left( \frac{3}{4} \right)^{n-1}}{1 - \frac{3}{4}}
\]

\[
= 10 + 60 \left( 1 - \left( \frac{3}{4} \right)^{n-1} \right).
\]

In particular, for \( n = 4 \), the answer is 44.6875 feet.

(c) As discussed above, the answer is \( 10 + 60 \left( 1 - \left( \frac{3}{4} \right)^{n-1} \right) \).

**Exercise 47**

(a) The formula for the height of the ball after \( t \) seconds is \( h - \frac{1}{2}gt^2 \). When the ball hits the ground, this expression must be 0, so \( t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{h}{16}} = \frac{1}{4} \sqrt{h} \).

(b) In exercise 46, the ball takes \( \frac{1}{4} \sqrt{10} \) seconds to hit the ground for the first time. Then, after hitting the ground for the \( n \)th time, the ball hits the height \( h_n = 10 \left( \frac{3}{4} \right)^n \) feet. The time it takes to bounce from the ground back to \( h_n \) feet height is the same as the time it takes to drop from that height to the ground, which is \( \frac{1}{4} \sqrt{h_n} \) seconds. Thus, the time between the \( n \)th and \((n + 1)\)th hit is \( \frac{1}{2} \sqrt{h_n} \) seconds. Hence the total travel time is

\[
\frac{1}{4} \sqrt{10} + \sum_{n=1}^{\infty} \frac{1}{2} \sqrt{h_n} = \frac{1}{4} \sqrt{10} + \sum_{n=1}^{\infty} \frac{1}{2} \sqrt{10} \left( \frac{3}{4} \right)^n
\]

\[
= \frac{1}{4} \sqrt{10} + \frac{1}{2} \sqrt{10} \cdot \frac{3}{4} \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n = \frac{1}{4} \sqrt{10} + \frac{1}{2} \sqrt{10} \cdot \frac{3}{4} \cdot \frac{1}{1 - \frac{3}{4}}
\]

**Book problems - section 9.3**

**Exercise 4** Set \( f(x) = \frac{1}{(x+2)^2} \). This function is continuous, positive, decreasing as \( x \) goes from 1 to \( \infty \). Thus, since the integral

\[
\int_1^\infty \frac{1}{(x+2)^2} \, dx = \lim_{b \to \infty} \left[ \frac{1}{x+2} \right]_1^b = \frac{1}{3}
\]

converges, the series \( \sum_{n=1}^{\infty} \frac{1}{(n+2)^2} \) also converge.
Exercise 6  Set \( f(x) = e^{-x} \). This function is continuous, positive, decreasing as \( x \) goes from 1 to \( \infty \). Thus, since the integral 
\[
\int_1^\infty e^{-x} \, dx = \lim_{b \to \infty} -e^{-x}|_1^b = e^{-1}
\]
converges, the series \( \sum_{n=1}^{\infty} e^{-n} \) also converge.

Exercise 10  The corresponding function is \( f(x) = x^2 \). This function is not decreasing as \( x \) goes from 1 to \( \infty \), so the integral test does not apply.

Exercise 12  The corresponding function is \( f(x) = e^{-x} \sin x \). This function does not stay positive as \( x \) goes from 1 to \( \infty \), so the integral test does not apply. (It is also not decreasing, since its value switches back and forth between positive and negative).

Exercise 14  Set \( f(x) = \frac{4}{2x+1} \). This function is continuous, positive, decreasing as \( x \) goes from 0 to \( \infty \). Thus, since the integral 
\[
\int_0^\infty \frac{4}{2x+1} \, dx = \lim_{b \to \infty} 2 \ln(2x+1)|_1^b = \infty
\]
diverges, the series \( \sum_{n=1}^{\infty} \frac{1}{(n+2)^2} \) also diverge.

Exercise 16  Set \( f(x) = \frac{2x}{1+x^4} \). This function is continuous, positive, decreasing as \( x \) goes from 1 to \( \infty \). (Continuity and positivity is clear; the fact that \( f(x) \) decreases can be checked by looking at the derivative 
\[
f'(x) = \frac{2(1+x^4) - 4x^3 \cdot 2x}{(1+x^4)^2} = \frac{2 - 6x^3}{(1+x^4)^2},
\]
which is negative for \( x \geq 1 \)).

Now we compute the integral \( \int_1^\infty \frac{2x}{1+x^4} \, dx \). Let \( u = x^2 \), then \( du = 2xdx \). Thus,
\[
\int_1^\infty \frac{2x}{1+x^4} \, dx = \int_1^\infty \frac{1}{1+u^2} \, du = \lim_{b \to \infty} \arctan(u)|_1^b = \frac{\pi}{2} - \arctan(1)
\]
is convergent. Therefore, by the integral test, the series \( \sum_{n=1}^{\infty} \frac{2n}{1+n^4} \) converge. Since convergence behaviour does not depend on where the index starts, the series \( \sum_{n=0}^{\infty} \frac{2n}{1+n^4} \) converge as well.

Exercise 18  We can use the divergence test here. Since 
\[
\lim_{n \to \infty} \frac{2n}{\sqrt{4+n^2}} = \lim_{n \to \infty} \frac{2}{\sqrt{\frac{4}{n^2} + 1}} = \frac{2}{\sqrt{0+1}} = 2 \neq 0,
\]
the series diverge.

Exercise 20  Set \( f(x) = \frac{4}{(2x+1)^3} \). This function is continuous, positive, decreasing as \( x \) goes from 1 to \( \infty \). Thus, since the integral 
\[
\int_1^\infty \frac{4}{(2x+1)^3} \, dx = \lim_{b \to \infty} -(2x+1)^{-2}|_1^b = \frac{1}{9}
\]
converges, the series \( \sum_{n=1}^{\infty} \frac{4}{(2n+1)^3} \) also converge.
Exercise 22  Set $f(x) = \frac{2}{1+4x^2}$. This function is continuous, positive, decreasing as $x$ goes from 0 to $\infty$. Thus, since the integral

$$\int_0^\infty \frac{2}{1+4x^2} \, dx = \lim_{b \to \infty} \arctan(2x)|_0^b = \frac{\pi}{2}$$

converges, the series $\sum_{n=0}^{\infty} \frac{2}{1+4n^2}$ also converge.

Exercise 28  Set $f(x) = \frac{\ln x}{x}$. This function is continuous, positive, decreasing as $x$ goes from 3 to $\infty$. (Continuity and positivity is clear; the fact that $f(x)$ decreases can be checked by looking at the derivative

$$f'(x) = \frac{1}{x} - \frac{\ln x}{x^2} = \frac{1 - \ln x}{x^2},$$

which is negative for $x \geq 3$). The integral $\int_3^\infty \frac{\ln x}{x} \, dx$ diverges, because $\frac{\ln x}{x} > \frac{1}{x}$ and $\int_3^\infty \frac{1}{x} \, dx$ diverges. Thus, by the integral test, the series $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverge. Since convergence behaviour does not depend on where the index starts, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverge as well.

Supplemental problem G

(a) 

$$I_0 = \int_0^{\frac{\pi}{2}} (\cos \theta)^0 \, d\theta = \int_0^{\frac{\pi}{2}} 1 \, d\theta = \frac{\pi}{2}; \quad I_1 = \int_0^{\frac{\pi}{2}} (\cos \theta)^1 \, d\theta = \int_0^{\frac{\pi}{2}} \cos \theta \, d\theta = \sin \theta|_0^{\frac{\pi}{2}} = 1.$$ 

(b) By the integration table,

$$\int (\cos \theta)^k \, d\theta = \frac{1}{k} (\cos \theta)^{k-1} \sin \theta + \frac{k-1}{k} \int (\cos \theta)^{k-2} \, d\theta.$$ 

Thus,

$$I_k = \int_0^{\frac{\pi}{2}} (\cos \theta)^k \, d\theta = \frac{1}{k} (\cos \theta)^{k-1} \sin \theta|_0^{\frac{\pi}{2}} + \frac{k-1}{k} \int_0^{\frac{\pi}{2}} (\cos \theta)^{k-2} \, d\theta.$$ 

Note that when $k \geq 2$, the value of $(\cos \theta)^{k-1} \sin \theta$ is 0 both when $\theta = 0$ and $\theta = \frac{\pi}{2}$. Thus,

$$I_k = \frac{k-1}{k} \int_0^{\frac{\pi}{2}} (\cos \theta)^{k-2} \, d\theta = \frac{k-1}{k} I_{k-2}.$$ 

(c) 

$$I_0 = \frac{\pi}{2}, \quad I_2 = \frac{2}{2} I_0 = \frac{\pi}{4}, \quad I_1 = 1, \quad I_4 = \frac{4}{4} I_2 = \frac{3\pi}{16}, \quad I_3 = \frac{3}{3} I_1 = \frac{2}{3}, \quad I_5 = \frac{5}{5} I_3 = \frac{8}{15}, \quad I_6 = \frac{6}{6} I_4 = \frac{5\pi}{32}, \quad I_7 = \frac{7}{7} I_5 = \frac{16}{35}.$$ 

(d) Decreasing.

(e) 

$$I_0 \cdot I_1 = \frac{\pi}{2}, \quad I_2 \cdot I_3 = \frac{\pi}{6}, \quad I_4 \cdot I_5 = \frac{\pi}{10}, \quad I_6 \cdot I_7 = \frac{\pi}{14}.$$ 

Generally,

$$I_{2n} \cdot I_{2n+1} = \frac{\pi}{2(2n+1)}.$$
Supplemental problem H

(a) Note that the rectangles in the figure below have area $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$ respectively, so $H_n$ is the total area of these $n$ rectangles. Meanwhile, $\ln(n + 1) = \int_1^{n+1} \frac{1}{x} \, dx$ is the area under the graph $y = \frac{1}{x}$ from 1 to $n + 1$. Since the function $\frac{1}{x}$ is decreasing, the graph is inside the rectangles, as shown in the figure. Thus, $H_n \geq \ln(n + 1)$. Since $\ln(n + 1) > \ln n$ for every $n$, we can conclude that $H_n > \ln n$. Thus, $g_n = H_n - \ln n$ is positive.

![Figure 1: An illustration for the case $n = 5$, showing why $H_5 > \int_1^6 \frac{1}{x} \, dx$.](image)

(b)

$$g_n - g_{n-1} = (H_n - \ln n) - (H_{n-1} - \ln(n - 1))$$

$$= (H_n - H_{n-1}) + \ln(n - 1) - \ln n$$

$$= \left(\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n-1} \frac{1}{k}\right) + \ln \left(\frac{n-1}{n}\right)$$

$$= \frac{1}{n} + \ln \left(1 - \frac{1}{n}\right).$$

(c) Here are the graphs. Clearly $\ln(1 - x)$ is below $-x$.
This inequality can also be seen without the graphs: In a previous homework, you showed that $e^x \geq 1 + x$ for every number $x$. Plugging in $\ln(1 - x)$ gives $e^{\ln(1-x)} \geq 1 + \ln(1 - x)$, which simplifies to $1 - x \geq 1 + \ln(1 - x)$, so $-x \geq \ln(1 - x)$.

(d) We just showed that $\ln(1 - x) < -x$ for every number $x$. Plugging in $\frac{1}{n}$ gives $\ln(1 - \frac{1}{n}) < -\frac{1}{n}$. Thus, by part (b), $g_n - g_{n-1} = \frac{1}{n} + \ln(1 - \frac{1}{n}) < \frac{1}{n} - \frac{1}{n} = 0$. Thus, $g_n < g_{n-1}$, so the sequence $g_n$ is decreasing.

(e) We just showed that the sequence $g_n$ is decreasing. By part (a), this sequence is also positive, so it is bounded below by 0. Any decreasing sequence that is bounded below is convergent.