homework 3 solutions.

Problem 1. Decide if the following statements are true or false and explain your answers:

a. You can tell if a sequence converges by looking at the first million terms.

Solution: No matter how promising the first million terms may look like, after that the series may “decide”, for example, to stop changing. The sum \( \sum_{n=10^6}^{\infty} a_{10^6} \) is infinite (as long as \( a_{10^6} \neq 0 \)), and:

\[
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{10^6} a_n + \sum_{n=10^6}^{\infty} a_n = \text{Infinite}.
\]

Thus the statement is FALSE.

b. If the terms of a convergent sequence are all > 0, then the limit of the sequence will also be > 0.

Solution: Statement is FALSE. Remember the example of \( a_n = \frac{1}{n} > 0 \) with the limit EQUAL to zero. What is TRUE, is that for converging sequence

\[
\text{if } a_n > 0 \text{ for every } n, \text{ then } \lim_{n \to \infty} a_n \geq 0.
\]

c. If a sequence of positive terms is not bounded, then the sequence must contain a term that is \( \geq 100^{100} \).

Solution: We shall argue by what is called contrapositive. The statement is TRUE if from negation of the conclusion follows the negation of the assumption.

The negation of:

the sequence \( \{a_n\} \) must contain a term that is \( \geq 100^{100} \).

is:

every term of the sequence \( \{a_n\} \) is \( < 100^{100} \).

Given that negation, and that \( \{a_n\} \) is a sequence of positive terms, we have that for any \( n \):

\[
0 < a_n < 100^{100}
\]

which means that \( \{a_n\} \) is bounded, which is a negation of the assumption.

Therefore the statement is TRUE.
d. If a sequence of positive terms is *not* bounded, then the sequence must contain *infinitely many* terms that are $\geq 100^{100}$.

**Solution:** Similarly like before, by contrapositive we show that the statement is TRUE. The negation of:

the sequence $\{a_n\}$ must contain infinitely many terms that are $\geq 100^{100}$.

is:

there are finitely many terms of the sequence $\{a_n\}$ that are $\geq 100^{100}$.

Let $M$ be the maximum of those finitely many terms that are $\geq 100^{100}$, or $M = 100^{100}$, if there are no such terms. Then for any $n$:

$$0 < a_n < M$$

which means that $\{a_n\}$ is bounded, which is a negation of the assumption. Therefore the statement is TRUE.

e. If a convergent sequence $\{a_n\}$ satisfies $0 \leq a_n \leq 1$ for every index $n$, then we will have $0 \leq \lim_{n \to \infty} (a_n) \leq 1$ as well.

**Solution:** The statement is TRUE based on two-sided version of part (b).

f. A monotone sequence cannot have both positive and negative terms.

**Solution:** The statement is FALSE. As an example, put $a_n = \frac{1}{2} - \frac{1}{n}$. This sequence starts out with $a_1 = -\frac{1}{2}$ and $a_2 = 0$, but then all other terms are positive. It is easy to see that this sequence is increasing.

True is the statement that:

A monotone sequence cannot have *infinite number* of both positive and negative terms.

but it is NOT what we asked in this problem.

**Problem 2.** Explain what is wrong with the following statements:

a. The geometric sequence $4, 1, \frac{1}{4}, \frac{1}{16}, \ldots$ converges to $\frac{4}{1-\frac{1}{4}} = \frac{16}{3}$.

**Solution:** Clearly the author of this statement is confused about the difference between a sequence and series. The SEQUENCE $a_n = \left(\frac{1}{4}\right)^{n-2}$ converges to ZERO. The proposed answer is the limit of the SERIES $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n-2}$.
b. The geometric series $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$ converges to $\frac{1}{1 - (-\frac{3}{2})} = \frac{2}{5}$.

Solution: Ooops, somebody forgot the check the assumptions! The formula:

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1 - q}$$

is valid ONLY when $|q| < 1$. In the statement of this problems $q = -\frac{3}{2}$.

c. The following geometric series is convergent: $0.00001 + 0.0001 + 0.001 + \cdots$.

Solution: The ratio of this geometric series os $r = \frac{0.0001}{0.00001} = 10$. Since this $r$ does NOT satisfy $-1 < r < 1$, the series does not converge.

Problem 3. Suppose that $\lim_{k \to \infty} (a_k) = 1$. Explain why the series $\sum_{k=1}^{\infty} a_k$ diverges.

Solution: From the definition of the limit it follows that no matter how small number $s > 0$ we take, then all but finitely many terms of $a_k$ will fit into $[1 - s, 1 + s]$. You can think of $s$ as “thickening” the plain number 1 into an interval. On the picture below we drew interval with $s = 1/2$. What really matters is that we want the “thickened” interval to stay away from zero. Thus, for all $k > M$ ($M$ may be big), we have:

$$1/2 < a_k < 3/2.$$  

To prove divergence, use the first inequality, to show by direct comparison:

$$\sum_{k=M}^{\infty} 1/2 < \sum_{k=M}^{\infty} a_k$$

Infinite sum of – no matter how small – numbers is infinite, hence $\sum_{k=M}^{\infty} a_k$ diverges. The original series $\sum_{k=1}^{\infty} a_k$ differs from divergent series by FINITELY many terms, hence that series diverges as well.
Problem 4. Draw a graph that clearly illustrates the inequality \[ \sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{1}{x} \, dx \] for \( n = 5 \).

![Graph of f(x) = 1/x](image)

**Solution:** On the picture above, each rectangle has an area equal to one of the terms of the series. The green rectangles cover the area below the graph of \( 1/x \) on the interval \([1, 6]\), and that area is equal to the integral \( \int_{1}^{6} \frac{1}{x} \, dx \).

For the sake of showing you comparison between graph of \( 1/x \) and \( 1/x^2 \) in the next problem, the \( y- \) axis has been extended three-fold. Thus the first rectangle – which should be a square – does not look like one.

Problem 5. Draw a graph that clearly illustrates the inequality \[ \sum_{k=1}^{n} \frac{1}{k^2} \leq 1 + \int_{1}^{n} \frac{1}{x^2} \, dx \] for \( n = 5 \).

![Graph of f(x) = 1/x^2](image)

**Solution:** Now we need to fit the rectangles underneath the graph of \( 1/x^2 \). To fit them all, we need to use the graph on \([0, \infty)\). Unfortunately \( \int_{0}^{\infty} \frac{1}{x^2} \, dx \) diverges because of the “problems” of \( 1/x^2 \) near \( x = 0 \). So we need to toss the first rectangle out (the rectangle marked in red on the graph). The area of that rectangle is \( 1 \times \frac{1}{1^2} = 1 \), which is the “extra” term that we see on the right hand side of inequality that we illustrate. The remaining four terms of the series fit underneath the graph of \( 1/x^2 \) on \([1, 5]\), thus are bounded from above by the integral \( \int_{1}^{5} \frac{1}{x^2} \, dx \).
Problem 6. For each of the sequences below, (i) find a closed formula for the terms (no \( \sum \) allowed in your formula!), (ii) decide whether it is convergent (and find its limit if so), (iii) decide whether it is bounded, (iv) decide whether it is monotone, and (v) decide whether it is geometric.

a. 1, −2, 3, −4, 5, −6, 7, −8, 9, −10, ...

Solution: In our definition of formulas we assume that the first element listed is \( a_1 \) (in principle we could call it \( a_0 \), or any other index).

- Sequence has formula: \( a_n = (-1)^{n-1} n \). Of course other correct answers include:
  \[ a_n = (-1)^{n+1} n \quad \text{or} \quad a_n = -(-1)^n n \quad \text{or} \quad a_n = -\cos(\pi n) \cdot n. \]

- This sequence diverges (even terms approach \( -\infty \), while odd terms approach \( +\infty \), so even in terms of \( \pm \infty \) the limit does not exist).
- \( \{a_n\} \) is not bounded: neither from below, nor from the top.
- \( \{a_n\} \) is not monotone, although it contains two monotone sub-sequences: the increasing sequence \( b_n = a_{2n-1} \), and decreasing sequence \( c_n = a_{2n} \).

I know, the term “subsequence” has not been defined in this course, but take it at an intuitive level, as “part of the big sequence”.

- \( \{a_n\} \) is not geometric: The “candidates” for quotient of geometric sequence would be:
  \[ \frac{a_2}{a_1} = -2, \quad \text{and} \quad \frac{a_3}{a_2} = -\frac{3}{2}. \]

Since those two are different, the sequence is NOT geometric.

b. \( \frac{1}{2 \cdot 1}, \frac{1}{3 \cdot 2}, \frac{1}{4 \cdot 3}, \frac{1}{5 \cdot 4}, \frac{1}{6 \cdot 5}, \ldots \)

Solution:

- Sequence has formula: \( a_n = \frac{1}{n \cdot (n+1)} \).
- This sequence converges to zero.
- In the next “bullet” we show that \( \{a_n\} \) is decreasing. As it is also converging to zero, we have:
  \[ 0 \leq a_n \leq a_1 = \frac{1}{2 \cdot 1} \]

so the sequence is bounded.

- \( \{a_n\} \) is decreasing, as \( n \cdot (n+1) < (n+1) \cdot (n+2) \), so \( \frac{1}{n \cdot (n+1)} > \frac{1}{(n+1) \cdot (n+2)} \) for every \( n \).

- \( \{a_n\} \) is not geometric:
  \[ \frac{a_2}{a_1} = \frac{1}{3}, \quad \text{and} \quad \frac{a_3}{a_2} = \frac{1}{2}. \]

Since those two are different, the sequence is NOT geometric.
c. $1, -1, 1, -1, 1, -1, 1, -1, \ldots$

**Solution:**
- Sequence has formula: $a_n = (-1)^{n-1}$.
- This sequence diverges (the limit does not exist).
- $\{a_n\}$ is bounded: $-1 \leq (-1)^n \leq 1$.
- $\{a_n\}$ is not monotone.
- Test for $\{a_n\}$ being geometric gives:
  
  \[
  \frac{a_2}{a_1} = -1, \quad \text{and} \quad \frac{a_3}{a_2} = -1.
  \]

Now it is visible, that the sequence IS geometric with $r = -1$.

d. $\frac{5}{2}, \frac{5}{\sqrt{2}}, 5, \ldots$

**Solution:**
- Sequence has formula: $a_n = \frac{5}{2} \left(\sqrt{2}\right)^{n-1}$.
- This sequence diverges, as in the last “bullet” we show that it is geometric with $r = \sqrt{2} > 1$.
- $\{a_n\}$ is not bounded for the same reason as above.
- $\{a_n\}$ is increasing, as above.
- It is clear from the formula that $\{a_n\}$ is geometric with $r = \sqrt{2}$.

e. $1, 1 - \frac{1}{3}, 1 - \frac{1}{3} + \frac{1}{9}, 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27}, \ldots$

**Solution:**
- This time – behind the scene – there is a geometric SEQUENCE $b_n = \left(\frac{-1}{3}\right)^{n-1}$, and $a_n$ is the sequence of partial sums of $\{b_n\}$. That means:
  
  \[
  a_1 = b_1, \quad a_2 = b_1 + b_2, \quad a_3 = b_1 + b_2 + b_3, \quad \text{etc}...
  \]

The formula for partial sums for geometric series, applied for $\left(\frac{-1}{3}\right)^{n-1}$ gives the compact formula:

\[
a_n = \frac{1 - \left(\frac{-1}{3}\right) \left(\frac{-1}{3}\right)^{n-1}}{1 - \left(\frac{-1}{3}\right)} = \frac{3}{4} \cdot \left(1 - \left(\frac{-1}{3}\right)^n\right).
\]

- From the formula it is evident, that this sequence converges to $\frac{3}{4}$.
- As a converging sequence, $\{a_n\}$ is bounded.
- $\{a_n\}$ is not geometric:
  
  \[
  \frac{a_2}{a_1} = \frac{2}{3}, \quad \text{and} \quad \frac{a_3}{a_2} = \frac{7}{9} = \frac{7}{6}.
  \]

Since those two are different, the sequence is NOT geometric.
Solution:
- Sequence has formula: $a_n = 2^{1/n}$.
- This sequence converges to 
  \[ \lim_{n \to \infty} \frac{1}{n} = 2^0 = 1. \]
- As a converging sequence, \( \{a_n\} \) is bounded.
- \( \{a_n\} \) is decreasing, as \( \frac{1}{n} \) is a decreasing sequence.
- \( \{a_n\} \) is not geometric: The “candidates” for quotient of geometric sequence would be:
  \[ \frac{a_2}{a_1} = \frac{1}{\sqrt{2}}, \quad \text{and} \quad \frac{a_3}{a_2} = \frac{1}{\sqrt{2}}. \]
Since those two are different, the sequence is NOT geometric.

Problem 7. The sequence \( \left\{ \frac{4n + (-1)^n \cdot 5}{5n + 1} \right\}_{n=0}^\infty \) converges.

a. Explain why you cannot apply L’Hôpital’s rule to evaluate the limit of this sequence, even though the numerator and denominator both tend to \( +\infty \).

Solution: For this problem we mostly expected a solution of the sort:
If we declare that \((1)^n = (-1)^x\), then this formula is not defined
e.g. for \(x = 1/2\) (that is, we can not take square root for negative number.)
You get full credit for that kind of answer. Formally showing one failed attempt to define \( f(x) \) for which \( f(n) = a_n \) does not prove that any other attempt will fail as well. Let me show one more attempt:
Let \( f(x) = \frac{4x + 5 \cos(\pi x)}{5x + 1} \). Again we have \( f(n) = \frac{4n + 5(-1)^n}{5n + 1} \), and this time both numerator and denominator are differentiable functions approaching infinity. Unfortunately attempt to use l’Hopital rule gives:
\[ \lim_{x \to \infty} \frac{4 - 5\pi \sin(\pi x)}{5} \]
which limit does not exist. It is a nice example to keep in mind for the importance of the conclusion of the statement of l’Hopital’s rule:
“...provided that the limit on the right-hand side exists.”
It was NOT an intention of this problem, and it is involved to show that for any two functions \( f(x) \) and \( g(x) \) such that \( f(n)/g(n) = \frac{4n + 5(-1)^n}{5n + 1} \) one can not apply l’Hopital rule.
b. Demonstrate how to find the limit of the sequence using another method.

Solution: Since \(-1 \leq (-1)^n \leq 1\) we can estimate arbitrary element of the sequence:

\[
\frac{4n - 5}{5n + 1} \leq \frac{4n + 5 \cdot (-1)^n}{5n + 1} \leq \frac{4n + 5}{5n + 1}
\]

The limits of fractions on both sides of inequalities converge to \(\frac{4}{5}\), so the limit in the middle must also converge to \(\frac{4}{5}\).

Problem 8. Each of the sums/series below are geometric. For each, (i) determine the value of \(r\) (the ratio between terms), (ii) write the sum in \(\sum\) notation, and (iii) evaluate the sum using the geometric sum formula, and write your answer as a whole number or a fraction in lowest terms.

General comment: For all problems below we are going to use the partial sum formula for geometric sequence, in the form:

\[
a_1 + a_2 + \cdots + a_{\text{whatever}} = \frac{[\text{first term}] - r \cdot [\text{last term}]}{1 - r}
\]

where \(r = \frac{a_2}{a_1}\) is the quotient of the sequence. We trust the statement

Each of the sums/series below are geometric.

but it is good idea to check if \(\frac{a_3}{a_2}\) and \(\frac{a_4}{a_3}\) also give the same \(r\).

a. \(1 - 2 + 4 - 8 + 16 - 32 + \cdots + 1024\)

Solution:

\[r = \frac{a_2}{a_1} = -2\]

For sum notation, the hardest problem is to establish the upper limit of the sum. Here we recognize 1024 as \((-2)^{10}\), so this is the \(n = 10\) term of the sequence:

\[
\sum_{n=0}^{10} (-2)^n = \frac{1 - (-2) \cdot 1024}{1 - (-2)} = \frac{2049}{3} = 683.
\]
b. \(60 + 20 + \frac{20}{3} + \frac{20}{9} + \cdots + \frac{20}{729}\)

**Solution:**

\[
r = \frac{a_2}{a_1} = \frac{20}{60} = \frac{1}{3}.
\]

To write in sum notation, we need to recognize 729 as \(3^6\). So

\[
\sum_{n=-1}^{6} 20 \cdot \left(\frac{1}{3}\right)^n = 60 - \frac{(\frac{1}{3})^{20}}{1 - \frac{1}{3}} = \frac{180 - \frac{20}{729}}{3 - 1} = \frac{65600}{729}.
\]

c. \(0.1 + 0.01 + 0.001 + \cdots\)

**Solution:**

\[
r = \frac{a_2}{a_1} = \frac{0.01}{0.1} = 0.1.
\]

Now this is infinite series, so no problem with establishing the index of the last term:

\[
\sum_{n=1}^{\infty} (0.1)^n = 0.1 \cdot \frac{1}{1 - 0.1} = \frac{1}{9}.
\]

d. \(0.01 + 0.0001 + 0.000001 + \cdots\)

**Solution:**

\[
r = \frac{a_2}{a_1} = \frac{0.0001}{0.01} = 0.01.
\]

And:

\[
\sum_{n=1}^{\infty} (0.01)^n = \frac{1}{100} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{1}{99}.
\]

e. \(1 + \frac{5}{3} + \frac{25}{9} + \cdots\)

**Solution:**

\[
r = \frac{a_2}{a_1} = \frac{\frac{5}{3}}{1} = \frac{5}{3}.
\]

And:

\[
\sum_{n=0}^{\infty} \left(\frac{5}{3}\right)^n \text{ which diverges, since } \frac{5}{3} > 1.
\]

f. \(2 \cdot 3 + \frac{2}{3} + \frac{2}{3^2} + \cdots\)

**Solution:**

\[
r = \frac{a_2}{a_1} = \frac{\frac{2}{3}}{2 \cdot 3} = \frac{1}{9}.
\]

So:

\[
\sum_{n=0}^{\infty} 6 \cdot \left(\frac{1}{9}\right)^n = 6 \cdot \frac{1}{1 - \frac{1}{9}} = \frac{27}{4}.
\]
Problem 9. Write the repeating decimal $0.\overline{73} = 0.737373\ldots$ as a fraction in lowest terms.

**Solution:** We have:

$$0.\overline{73} = 0.73 + 0.73 \cdot \frac{1}{100} + 0.73 \cdot \left( \frac{1}{100} \right)^2 + \cdots$$

Hence $0.\overline{73}$ is the sum of geometric series with $a_0 = 0.73$ and $r = \frac{1}{100}$. The sum of such series is:

$$0.\overline{73} = \frac{73}{100} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{73}{99}$$

Since 73 is a prime number, this fraction can not be simplified.


a. Exercise 4: use the integral test to decide whether the series $\sum_{n=1}^{\infty} \frac{1}{(n + 2)^2}$ converges.

**Solution:** Use the function

$$f(x) = \frac{1}{(x + 2)^2}$$

As $(x + 2)^2$ is increasing on $[-2, \infty)$, the reciprocal $\frac{1}{(x + 2)^2}$ is decreasing on the same interval. Clearly $f(x) > 0$, and it is defined on $[1, \infty)$, so we have green light to use the integral test:

$$\int_1^{\infty} \frac{1}{(x + 2)^2} \, dx = -\frac{1}{x + 2}\bigg|_1^{\infty} = \frac{1}{3}, \text{ thus the integral converges.}$$

Thus the original series converges.

In fact for this problem one does not need to compute the value of the integral: all that is needed is noticing that $f(x) \propto \frac{1}{x^2}$, and the latest function has converging integral on $[1, \infty)$. You need the value of the integral only when you are asked to estimate the value of the series.

b. Exercise 6: use the integral test to decide whether the series $\sum_{n=1}^{\infty} \frac{1}{e^n}$ converges.

**Solution:** It is evident that $f(x) = \frac{1}{e^x} = e^{-x}$ is defined, decreasing and positive. Hence we can use the integral test:

$$\int_1^{\infty} e^{-x} \, dx = -e^{-x}\bigg|_1^{\infty} = \frac{1}{e}, \text{ thus the integral converges.}$$

Thus the original series converges.
c. Exercise 8: Use comparison with \( \int_1^\infty x^{-3} \, dx \) to show that \( \sum_{n=2}^{\infty} \frac{1}{n^3} \) converges to a number that is less than or equal to 1/2.

\[
f(x) = \frac{1}{x^3}
\]

\[
\begin{align*}
\sum_{n=2}^{\infty} \frac{1}{n^3} \leq \int_1^{\infty} \frac{1}{x^3} \, dx &= -\frac{1}{2} x^{-2} \bigg|_1^{\infty} = \frac{1}{2} \\
\end{align*}
\]

In fact since there is a “slack” between green boxes and the region below the graph of \( f(x) \), the inequality is strict, i.e. \( \sum_{n=2}^{\infty} \frac{1}{n^3} < \frac{1}{2} \).

**Problem 11.** The Fibonacci sequence is defined by \( F_0 = 0, F_1 = 1 \), and \( F_{n+1} = F_n + F_{n-1} \) for \( n \geq 1 \). That is, each term is the sum of the previous two terms.

a. Write down the first 13 terms of the Fibonacci sequence.

**Solution:** First 13 terms of Fibonacci sequence:

\[
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144.
\]
b. Though the Fibonacci sequence is *not literally* geometric, the limit

\[ \varphi = \lim_{n \to \infty} \left( \frac{F_{n+1}}{F_n} \right) \]

exists and is positive. Using the fact that this limit exists, find an exact value for \( \varphi \).

**Solution:** If

\[ \varphi = \lim_{n \to \infty} \left( \frac{F_{n+1}}{F_n} \right) \]

then

\[ \frac{1}{\varphi} = \lim_{n \to \infty} \left( \frac{F_n}{F_{n+1}} \right) = \lim_{n \to \infty} \left( \frac{F_{n-1}}{F_n} \right) \]

where in the last equality we have just shifted the index from \( n \) to \( n - 1 \). Shifting index does not change the limit.

Using the recurrence relation we have:

\[ \varphi = \lim_{n \to \infty} \left( \frac{F_{n+1}}{F_n} \right) = \lim_{n \to \infty} \left( \frac{F_n + F_{n-1}}{F_n} \right) = \lim_{n \to \infty} \left( 1 + \frac{F_{n-1}}{F_n} \right) = 1 + \frac{1}{\varphi} . \]

Multiply both sides of that formula by \( \varphi \) to get that \( \varphi \) is a solution to the quadratic equation:

\[ \varphi^2 = \varphi + 1 \]

This equation has two roots:

\[ \varphi_1 = \frac{1 + \sqrt{5}}{2} \quad \varphi_2 = \frac{1 - \sqrt{5}}{2} \]

Since for every \( n, F_n > 0 \), so the limit \( \varphi \) can not be negative, hence \( \varphi = \frac{1 + \sqrt{5}}{2} \).

c. By (b), \( F_{n+1} \approx \varphi \cdot F_n \) for large indices \( n \), so \( F_n \approx \varphi^n \) as \( n \to \infty \). More precisely, we have the formula

\[ F_n = \left[ \frac{\varphi^n}{\sqrt{5}} \right] \]

where \([y]\) means "\( y \) rounded to the nearest integer." Use this (and a calculator with an exponentiation function) to find \( F_{20} \), the 20th Fibonacci number.

**Solution:** \([6765.000029563931874] = 6765\).
d. Suppose that PLANET R is home to robots and rabbits. If the number of robots after $n$ weeks is $2^n$ and the number of rabbits after $n$ weeks is $F_n$, then who will eventually dominate the planet? That is, one of

$$\lim_{n \to \infty} \left( \frac{\text{robots}}{\text{robots} + \text{rabbits}} \right) \quad \text{or} \quad \lim_{n \to \infty} \left( \frac{\text{rabbits}}{\text{robots} + \text{rabbits}} \right)$$

will be equal to 1. Which, and why? Alternatively, you can show (by taking a limit) that either robots $\prec$ rabbits or robots $\succ$ rabbits.

**Solution:** Since $F_n \sim \varphi^n$ and $\varphi = \frac{1 + \sqrt{5}}{2} < \frac{1 + 3}{2} = 2$, then the growth of robots given by $2^n$ dominates the growth of rabbits. Thus – sadly – the robots will dominate.

**Problem 12.** For this problem $I_k = \int_0^{\pi/2} (\cos \theta)^k \, d\theta$.

a. Show that $I_0 = \frac{\pi}{2}$ and that $I_1 = 1$.

**Solution:**

$$I_0 = \int_0^{\pi/2} (\cos \theta)^0 \, d\theta = \int_0^{\pi/2} 1 \, d\theta = \frac{\pi}{2}.$$

$$I_1 = \int_0^{\pi/2} (\cos \theta)^1 \, d\theta = -\sin \theta|_0^{\pi/2} = 1.$$

b. Using the book’s integration table, show that the sequence $\{I_k\}$ satisfies the “two steps back” recurrence $I_k = \frac{k-1}{k} I_{k-2}$ (so long as $k \geq 2$).

**Solution:** (b) Putting the limits from 0 to $\pi/2$ into the formula 18 from the back cover of the text, we get:

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x \bigg|_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x \, dx$$

Since $\sin(0) = 0$ and $\cos(\pi/2) = 0$, the middle term cancels out. The remaining part gives the desired formula:

$$I_n = \frac{n-1}{n} I_{n-2}.$$
c. Use (a.) and (b.) to compute the values of \( I_0, I_1, I_2, \ldots, I_7 \). You should express your answers either as nice fractions (for the odd-indexed terms), or as nice fraction multiples of \( \frac{\pi}{2} \) (for the even-indexed terms).

**Solution:** Use the formula \( I_k = \frac{k-1}{k} I_{k-2} \) we get:

\[
I_0 = \frac{\pi}{2} \quad \Rightarrow \quad I_2 = \frac{1}{2} \cdot \frac{\pi}{2} \quad \Rightarrow \quad I_4 = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \Rightarrow \quad I_6 = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}.
\]

and:

\[
I_1 = 1 \quad \Rightarrow \quad I_3 = \frac{2}{3} \quad \Rightarrow \quad I_5 = \frac{4}{5} \cdot \frac{2}{3} \quad \Rightarrow \quad I_7 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}.
\]

d. \( \{I_k\} \) is in fact a monotone sequence. Based on the values you found in (c.), is \( \{I_k\} \) increasing or decreasing?

**Solution:** Since \( I_k = \frac{k-1}{k} I_{k-2} = \left(1 - \frac{1}{k}\right) I_{k-2} \), it is easy to see that the sequence of even terms is decreasing and that the sequence of odd terms is increasing as well. Even though we know that \( I_0 > I_1 \), it is not immediate that the whole sequence is decreasing as well. But here we are told that the whole sequence is monotone, thus it must be decreasing.

e. Using the values you found in (c.), find a nice formula for \( I_k \cdot I_{k+1} \) \( (I_k \ * \text{times} \ * I_{k+1}) \) in terms of \( k \). Hint: It’s easier to see what the formula will be if you write each \( I_k \cdot I_{k+1} \) as a nice fraction multiple of \( \frac{\pi}{2} \).

**Solution:** Set \( c_k = I_k \cdot I_{k+1} \). Then:

\[
c_{k+1} = I_{k+1} \cdot I_{k+2} = I_{k+1} \cdot \frac{k+1}{k+2} I_k = c_k \cdot \frac{k+1}{k+2}.
\]

Using (from part (a)) that \( c_0 = \frac{\pi}{2} \) we get:

\[
c_k = \frac{k}{k+1} c_{k-1} = \frac{k}{k+1} \cdot \frac{k-1}{k} c_{k-2} = \frac{k}{k+1} \cdot \frac{k-1}{k} \cdot \frac{k-2}{k-1} \cdots \frac{1}{2} c_0.
\]

In this long product denominator of each fraction cancels numerator of proceeding fraction. Thus in the end:

\[
c_k = \frac{1}{k+1} c_0 = \frac{1}{k+1} \cdot \frac{\pi}{2}.
\]