Book Problems

• **Section 9.4:** 32, 34, 36*, 60, 64, 66, 68, 70, 72, 74, 78, 86

* For this problem, you will need to show (by carefully taking a limit) that \( \arcsin(\frac{1}{x}) \approx \frac{1}{x} \).

**Problem I: The Fibonacci sequence**

The Fibonacci sequence is defined by \( F_0 = 0, F_1 = 1, \) and \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2 \).

a. Though the Fibonacci sequence is not literally geometric, the limit

\[
\varphi = \lim_{n \to \infty} \left( \frac{F_{n+1}}{F_n} \right)
\]

exists and is positive. Find this limit.

*Hint 1: Use the recurrence formula for \( F_{n+1} \).*

*Hint 2: If \( \frac{F_{n+1}}{F_n} \to \varphi \) as \( n \to \infty \), then what is the limit of \( \frac{F_{n-1}}{F_n} \) as \( n \to \infty \)?

*Hint 3: Perhaps unexpectedly, you’re going to need the quadratic formula!*

b. By (a), \( F_{n+1} \approx \varphi \cdot F_n \) for large indices \( n \), so \( F_n \approx \varphi^n \) as \( n \to \infty \). More precisely, we have the exact, closed formula

\[
F_n = \left[ \frac{\varphi^n}{\sqrt{5}} \right]
\]

where \( [y] \) means “\( y \) rounded to the nearest integer.” Use this (and a calculator with an exponentiation function) to find \( F_{50} \), the 50th Fibonacci number.

c. Determine whether \( \sum_{n=1}^{\infty} \frac{1}{F_n} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \cdots \) converges or diverges.

*(Problems J and K are on the next two pages.)*
Problem J: Divergence of the harmonic series

In class we proved that the Harmonic series

\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \]

diverges by the integral test: it diverges because it is ≥ the divergent improper integral \( \int_{1}^{\infty} \frac{1}{x} \, dx \). However, it is not necessarily intuitively clear why the integral diverges, either (while, in contrast, \( \int_{1}^{\infty} \frac{1}{x^2} \, dx \) converges!)

Below are two more arguments for the divergence of the Harmonic series that may appeal to your intuition a bit better than the integral test argument. The parts you have to answer are in boldface.

a. Consider the following:

\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \left( \frac{1}{9} + \cdots + \frac{1}{16} \right) + \cdots \]

Every term in parentheses is \( \geq 1/2 \). Why? Use this observation to explain why the Harmonic Series diverges.

b. Here’s another argument using a method called “proof by contradiction” or the fancier Latin name “reductio ad absurdum.” We will start by assuming (falsely) that the harmonic series converges and we will use that hypothesis to deduce something bizarre and obviously false. This will mean our hypothesis was wrong in the first place, so the Harmonic Series must diverge. Here we go!

Suppose that the harmonic series converges to a real number we’ll call \( H \). Then \( H = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \). Grouping pairs of terms starting from \( 1/3 \),

\[ H = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} \right) + \left( \frac{1}{7} + \frac{1}{8} \right) + \cdots \]

\[ \geq 1/2 \quad \geq 1/3 \quad \geq 1/4 \]

Notice that we have the pattern \( \frac{1}{3} + \frac{1}{4} \geq \frac{1}{2} \), and \( \frac{1}{5} + \frac{1}{6} \geq \frac{1}{3} \), and etc. Why? Next, using comparison and rearranging terms,

\[ H \geq \frac{1}{2} + \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \right) \]

but this is absurd!! Why? What can we conclude about \( H \)?
Problem K: Gauss’ integral, Ep. 3

We are trying to show \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \). It’s been a long and arduous journey but it’s almost over! Here’s what we’ve proven so far:

I. For any integer \( n \geq 0 \), we have

\[ \sqrt{n} \cdot I_{2n+1} \leq \int_0^{\sqrt{n}} e^{-x^2} \, dx \leq \sqrt{n} \cdot I_{2n-2} \]

where \( I_k = \int_0^{\pi/2} (\cos \theta)^k \, d\theta \).

II. We know that the sequence \( \{I_k\} \) is decreasing.

III. The terms of the sequence \( \{I_k\} \) satisfy the following formula:

\[ I_k \cdot I_{k+1} = \frac{1}{k+1} \cdot \frac{\pi}{2} \]

And now, the finale: Squaring the inequality in (I.) yields

\[ n \cdot I_{2n+1}^2 \leq \left( \int_0^{\sqrt{n}} e^{-x^2} \, dx \right)^2 \leq n \cdot I_{2n-2}^2 \]

Since \( \{I_k\} \) is decreasing, we have \( I_{2n+1} \geq I_{2n+2} \) and \( I_{2n-2} \leq I_{2n-3} \) (we need \( n \geq 2 \) for this second inequality to even make sense).

a. Combine the various facts above to show that

\[ \frac{n}{2n+2} \cdot \frac{\pi}{2} \leq \left( \int_0^{\sqrt{n}} e^{-x^2} \, dx \right)^2 \leq \frac{n}{2n-2} \cdot \frac{\pi}{2} \]

b. Taking square roots of the above,

\[ \sqrt{\frac{n}{2n+2}} \cdot \frac{\pi}{2} \leq \int_0^{\sqrt{n}} e^{-x^2} \, dx \leq \sqrt{\frac{n}{2n-2}} \cdot \frac{\pi}{2} \]

Show that the limits of the expressions on the left and the right as \( n \to \infty \) are both equal to \( \frac{1}{2} \sqrt{\pi} \).

c. Finally, from the above, since \( \sqrt{n} \to \infty \) as \( n \to \infty \), we have \( \int_0^{\infty} e^{-x^2} \, dx = \frac{1}{2} \sqrt{\pi} \). Why may we conclude that \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \)?