Conceptual questions

1. Partario is attempting to determine whether \( \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) \) converges or diverges.

He believes this series converges and his argument is as follows:

i. We have \( \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{k=1}^{\infty} \frac{1}{k+1} \).

ii. Both \( \sum_{k=1}^{\infty} \frac{1}{k} \) and \( \sum_{k=1}^{\infty} \frac{1}{k+1} \) diverge.

iii. Therefore, \( \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \infty - \infty = 0 \).

iv. Since the series evaluates to a real number, the series converges.

Answer these questions about Partario’s argument and also the series:

a. Step (i) is already incorrect. The book gives circumstances under which we can split a series like this. Find that statement and remind Partario of when we’re allowed to use \( \sum (A \pm B) = \sum A \pm \sum B \) for infinite series.

b. Step (iii) is incorrect. Explain why.

c. Without using future parts of this problem, explain why the series must either converge to a positive real number or diverge to positive infinity.

d. The series does in fact converge. Give a correct limit comparison argument for why the series converges. Hint: Simplify the terms first.

e. This convergent series can actually be evaluated pretty easily! Explain why

\[
\sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}
\]

Hint: If you get stuck, try figuring out why the formula works for a small value of \( n \), like \( n = 3 \). Then, use the formula above to evaluate the infinite series.
2. Give examples of:

a. A convergent series \( \sum_{n=1}^{\infty} a_n \) such that \( \lim_{n \to 0} (a_n) = 0 \).

b. A divergent series \( \sum_{n=1}^{\infty} b_n \) such that \( \lim_{n \to 0} (b_n) = 0 \).

c. A convergent series \( \sum_{n=1}^{\infty} c_n \) of positive terms such that \( \sum_{n=1}^{\infty} \sqrt{c_n} \) diverges.

3. For each of the test/series pairs below, explain why that test cannot be applied to that series.

a. Integral test for \( \sum_{k=1}^{\infty} \frac{1 + (\sin k)^2}{k^2} \).

b. Integral test for \( \sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k} \).

c. Integral test for \( \sum_{k=0}^{\infty} [\sec k - 1] \).

d. Direct comparison test for \( \sum_{k=1}^{\infty} \frac{\cos k}{k} \).

**Routine problems**

Determine whether the following series converge or diverge. You may use any of the following methods/tests:

- Evaluation (rarely possible)
- Divergence test
- Integral test
- Geometric series test
- \( p \)-series test
- Limit comparison test
- Direct comparison test

Be sure to state which test(s) you are using in your argument, check that the conditions required by that test are true, and show the work required to implement that test.

4. a. \( \sum_{k=1}^{\infty} k \)  
    b. \( \sum_{k=1}^{\infty} (-1)^k \cdot k \)  
    c. \( \sum_{k=1}^{\infty} [(-1)^k + (-1)^{k+1}] \cdot k \)
5. a. $\sum_{k=4}^{\infty} \frac{1}{k^2 \ln k}$  
   b. $\sum_{k=4}^{\infty} \frac{1}{k(\ln k)^2}$  
   c. $\sum_{k=4}^{\infty} \left( \frac{\ln k}{k} \right)^2$

6. a. $\sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n(-3)^n}{(-4)^n(-5)^n}$  
   b. $\sum_{n=2}^{\infty} \left( \frac{1}{n} \right)^n$  
   c. $\sum_{n=2}^{\infty} n^{(-1)^n}$

7. a. $\sum_{n=0}^{\infty} \frac{1}{4n^2 + 1}$  
   b. $\sum_{n=1}^{\infty} \frac{n^2}{(\sqrt{n} + 1)^6}$  
   c. $\sum_{n=1}^{\infty} \frac{(3n+1)^3}{(4n+1)^4}$

8. a. $\sum_{n=0}^{\infty} \frac{(3n+1)^n}{4n+1}$  
   b. $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$  
   c. $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

**In-depth problems**

9. In class we proved that the harmonic series

\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots
\]

diverges by the *integral test*. In more detail, using a geometric argument, one can show that the value of the harmonic series must be $\geq \int_1^{\infty} \frac{1}{x} \, dx$, but this integral diverges to infinity. However, it is not necessarily *intuitively* clear why the integral diverges either. Below, we will outline another argument that shows that the harmonic series diverges, *without* using the integral test.

   a. Consider the following:

\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \left( \frac{1}{2} \right) + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \left( \frac{1}{9} + \cdots + \frac{1}{16} \right) + \cdots
\]

Every term in parentheses is $\geq 1/2$. Explain why.

   b. Explain how to use (a) to establish the inequality $\sum_{n=1}^{2m} \frac{1}{n} \geq 1 + \frac{m}{2}$.

   *For example, this means that* $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{1024} \geq 1 + \frac{10}{2} = 6$ (since $1024 = 2^{10}$). *The left hand side is difficult to compute (its exact value is a fraction where the numerator and denominator have > 400 digits), but it is approximately 7.51, which is bigger than 6, like we expect from the inequality.*

   c. Taking limits as $m \to \infty$, use (b) to explain why the harmonic series diverges.

10. (Gauss’ integral pt. 3 will be filled in here by AM of Saturday 2/09).