Exercises

For exercises 1 and 2, use direct comparison to determine whether the series converges or diverges.

1. \( \sum_{n=1}^{\infty} \frac{1}{3^n + 1} \)

2. \( \sum_{n=1}^{\infty} \frac{n + n \cdot (\sin n)^2}{n^3 + 1} \).

For exercises 3-6, use the ratio test to determine whether the series converges or diverges.

3. \( \sum_{n=1}^{\infty} \frac{n}{2^n} \)

4. \( \sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!} \)

5. \( \sum_{n=1}^{\infty} \frac{1}{r^n n!}, \ r > 0 \)

6. \( \sum_{n=0}^{\infty} \frac{2^n}{n^3 + 1} \)

For exercises 7-10, use limit comparison to determine whether the series converges or diverges.

7. \( \sum_{n=1}^{\infty} \frac{5n + 1}{3n^2} \)
8. \[\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)\] \(\text{Hint: compare with } \sum_{n=1}^{\infty} \frac{1}{n^2}\)

9. \[\sum_{n=10}^{\infty} \frac{4 \sin n + n}{n^2}\]

10. \[\sum_{n=-3}^{\infty} \frac{2^n}{3^n - 2}\]

**Problem I: The Fibonacci sequence**

The Fibonacci sequence is defined by \(F_0 = 0\), \(F_1 = 1\), and \(F_n = F_{n-1} + F_{n-2}\) for \(n \geq 2\).

a. Though the Fibonacci sequence is not literally geometric, the limit

\[\varphi = \lim_{n \to \infty} \left(\frac{F_{n+1}}{F_n}\right)\]

exists and is positive. Find this limit.

**Hint 1:** Remember that \(F_{n+1}\) is the sum of the two previous terms.

**Hint 2:** If \(\frac{F_{n+1}}{F_n} \to \varphi\) as \(n \to \infty\), then what is the limit of \(\frac{F_{n-1}}{F_n}\) as \(n \to \infty\)?

**Hint 3:** Perhaps unexpectedly, you’re going to need the quadratic formula!

b. By (a), \(F_{n+1} \approx \varphi \cdot F_n\) for large indices \(n\), so \(F_n \approx \varphi^n\) as \(n \to \infty\). More precisely, we have the exact, closed formula

\[F_n = \left[\frac{\varphi^n}{\sqrt{5}}\right]\]

where \([y]\) means “\(y\) rounded to the nearest integer.” Use this (and a calculator with an exponentiation function) to find \(F_{20}\), the 20th Fibonacci number.

c. Though he was not the first to study this sequence, it is named after the 12th century Italian mathematician Leonardo “Fibonacci” of Pisa. He discovered the sequence while imagining an idealized population growth model for rabbits. (Look it up, if you’re curious.)

Suppose that PLANET R is home to robots and rabbits. If the number of robots after \(n\) weeks is \(2^n\) and the number of rabbits after \(n\) weeks is \(F_n\), then who will eventually dominate the planet? That is, one of

\[\lim_{n \to \infty} \left(\frac{\text{robots}}{\text{robots} + \text{rabbits}}\right) \quad \text{or } \lim_{n \to \infty} \left(\frac{\text{rabbits}}{\text{robots} + \text{rabbits}}\right)\]

will be equal to 1. Which, and why?

d. Determine whether \(\sum_{n=1}^{\infty} \frac{1}{F_n} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \cdots\) converges or diverges.
Problem J: Stirling’s Approximation

In many fields (computer science especially) it is important to have a handle on how quickly the sequence of factorials \( n! \) grows. In class, we learned that

\[
\text{any exponential growth } r^n \prec n! \prec n^n
\]

However, we can find a much more precise estimate by comparing sums and integrals.

a. Let \( n \geq 1 \). Explain why \( \ln(n!) = \sum_{k=1}^{n} \ln(k) \).

b. Comparing sums and integrals (as in our explanation for the integral test!—except it’s important that \( \ln x \) is increasing and not decreasing), we have the inequality

\[
\int_1^n \ln x \, dx \leq \sum_{k=1}^{n} \ln(k) \leq \int_1^{n+1} \ln x \, dx
\]

Justify this inequality by drawing the relevant picture for \( n = 4 \), noting that the sum \( \sum_{k=1}^{4} \ln(k) \) is equal to the 4th left Riemann sum for \( \int_1^5 \ln x \, dx \) and also equal to the 3rd right Riemann sum for \( \int_1^5 \ln x \, dx \).

c. Combine (a.) and the integral formula for \( \int \ln x \, dx \) (in your textbook) to conclude that

\[
n \ln n - n + 1 \leq \ln(n!) \leq (n + 1) \ln(n + 1) - n
\]

d. Raising \( e \) to the inequality from (c.), and simplifying carefully, show that

\[
e \cdot \frac{n^n}{e^n} \leq n! \leq \frac{(n + 1)^{n+1}}{e^n}
\]

e. Show that \( \lim_{n \to \infty} \left( \frac{(n + 1)^{n+1}}{n^{n+1}} \right) = e \)

\[\text{Hint: } 1^\infty \text{ is an indeterminate form. You will need to use the special limit } \lim_{n \to \infty} (1 + \frac{1}{n})^n = e.\]

So for large \( n \), the upper bound on \( n! \) in (d) is \( \approx e \cdot \frac{n^{n+1}}{e^n} \). Replacing our upper bound with this approximation, we get \( e \cdot \frac{n^n}{e^n} \leq n! \leq n \left( e \cdot \frac{n^n}{e^n} \right) \) which is true for all \( n \geq 1 \) (note that the terms in the parentheses on left and right are the same).

So what’s the correct asymptotic for \( n!(??) \) It turns out the answer is in some sense “half-way” between the two sides of the inequality above:

\[
n! \asymp n^{1/2} \left( \frac{n^n}{e^n} \right) = n^{n+1/2} e^{-n}
\]

This approximation (and more precise versions of it) are called Stirling’s approximation for the factorial. There are a couple series in this class whose convergence can only be determined using Stirling’s approximation. For example:

f. Determine whether the series \( \sum_{n=0}^{\infty} \frac{(2n)!}{4^n \cdot (n!)^2} \) converges or diverges.

\[\text{Hint: The Ratio Test will be inconclusive. *Carefully* try limit comparison, using Stirling’s approximation.}\]