HOMEWORK 4 SOLUTIONS

BEN LIM

Section 9.4 problem 8
Since $3^n + 1 > 3^n$, it is clear we have the inequality

$$\sum_{n \geq 1} \frac{1}{3^n + 1} \leq \sum_{n \geq 1} \frac{1}{3^n}.$$ 

The sum on the right converges because it is just a geometric series with $r = 1/3$. By comparison, the sum on the left converges.

Section 9.4 problem 12
Since $\sin^2 n \leq 1$, we have

$$\sum_{n=1}^{\infty} \frac{n \sin^2 n}{n^3 + 1} \leq \sum_{n=1}^{\infty} \frac{n}{n^3 + 1} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$ 

The rightmost series converges because it is a $p$-series with $p = 2$. Therefore by comparison the original series converges.

Section 9.4 problem 14
Define

$$a_n := \frac{n}{2^n}.$$ 

We have

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n+1}{n} \cdot \frac{2^n}{n^{n+1}} = \frac{1}{2} \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = \frac{1}{2}.$$ 

Therefore by the ratio test, $\sum_{n=1}^{\infty} a_n$ converges.

Section 9.4 problem 16 Define $a_n = \frac{(n!)^2}{(2n)!}$. We have

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)!^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \lim_{n \to \infty} \frac{(n+1)!}{(2n+2)(2n+1)!} \cdot \frac{(2n)!}{(n!)^2} = \lim_{n \to \infty} \frac{n^2 + 2n + 1}{4n^2 + 6n + 1} = \frac{1}{4}.$$ 

Therefore by the ratio test, $\sum_{n=1}^{\infty} a_n$ converges.

Section 9.4 problem 18
Define

$$a_n := \frac{1}{r^n n!}.$$
We have
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{r^{n+1}(n+1)!} \cdot \frac{r^n n!}{1} = \lim_{n \to \infty} \frac{1}{r} \cdot \frac{1}{n+1} = 0.
\]

Therefore by the ratio test, \( \sum_{n=1}^{\infty} a_n \) converges.

**Section 9.4 problem 20**

Define
\[
a_n := \frac{2^n}{n^3 + 1}.
\]

We have
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2^{n+1}}{(n+1)^3 + 1} \cdot \frac{n^3 + 1}{2^n} = \lim_{n \to \infty} 2 \cdot \frac{(n+1)^3 + 1}{n^3 + 1} = \lim_{n \to \infty} 2 \cdot \frac{(1 + \frac{1}{n})^3 + \frac{1}{n^3}}{1 + \frac{1}{n^3}} = 2.
\]

Therefore by the ratio test, \( \sum_{n=1}^{\infty} a_n \) diverges.

**Section 9.4 problem 38**

Define
\[
a_n = \frac{5n + 1}{3n^2},
b_n := \frac{1}{n}.
\]

We will show that \( a_n \asymp b_n \) and since the series for \( b_n \) diverges, the series for \( a_n \) will diverge too by limit comparison. We have
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{(5n + 1)n}{3n^2} = \lim_{n \to \infty} \frac{5 + \frac{1}{n}}{3} = \frac{5}{3}.
\]

This shows that \( a_n \asymp b_n \) as claimed.

**Section 9.4 problem 40**

Define
\[
a_n = 1 - \cos \frac{1}{n},
b_n := \frac{1}{n^2}.
\]

We will show that \( a_n \asymp b_n \) and since the series for \( b_n \) converges, the series for \( a_n \) will converge too by limit comparison. We have
This shows that \( a_n \asymp b_n \) as claimed.

**Section 9.4 problem 42**

Define

\[
a_n := \frac{n + 1}{n^2 + 2}.
\]

How do we know which sequence to compare \( a_n \) to? Observe that \( a_n \) is the ratio of two polynomials in \( n \). Therefore, as we have discussed in class and section the sequence to compare \( a_n \) with is just \( n^k \), where

\[
k = \text{(highest power of } n \text{ in numerator - highest power of } n \text{ in denominator)} = -1.
\]

Therefore, we should compare \( a_n \) with

\[
b_n := \frac{1}{n}.
\]

We have

\[
limit_{n \to \infty} \frac{a_n}{b_n} = limit_{n \to \infty} \frac{(n + 1)n}{n^2 + 2}
\]

\[
= limit_{n \to \infty} \frac{1 + \frac{1}{n}}{1 + \frac{2}{n^2}}
\]

\[
= 1.
\]

Therefore \( a_n \asymp b_n \) and since the series for \( b_n \) diverges, by limit comparison so does the one for \( a_n \).

**Section 9.4 problem 44**

Define

\[
a_n := \frac{2^n}{3^n - 1}.
\]

When \( n \) is very very big, the 1 in the denominator is inconsequential and this gives us an indication that we should compare \( a_n \) with

\[
b_n := \frac{2^n}{3^n}.
\]

We have

\[
limit_{n \to \infty} \frac{a_n}{b_n} = limit_{n \to \infty} \frac{2^n}{3^n - 1} \cdot \frac{3^n}{2^n}
\]

\[
= limit_{n \to \infty} \frac{3^n}{3^n - 1}
\]

\[
= limit_{n \to \infty} \frac{1}{1 - \frac{1}{3^n}}
\]

\[
= 1.
\]
Therefore \(a_n \asymp b_n\) and since the series for \(b_n\) converges, by limit comparison so does the one for \(a_n\).

**Problem I**

(a) Using the relation \(F_{n+1} = F_n + F_{n-1}\), we get \(\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}\). Therefore
\[
\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = 1 + \lim_{n \to \infty} \frac{F_{n-1}}{F_n}. 
\]

However, notice that
\[
\lim_{n \to \infty} \frac{F_{n-1}}{F_n} = \lim_{n \to \infty} \frac{F_n}{F_{n+1}}
\]
and therefore writing
\[
\varphi := \lim_{n \to \infty} \frac{F_{n+1}}{F_n},
\]
we get from (*) the relation
\[
\varphi = 1 + \frac{1}{\varphi}
\]
and therefore
\[
\varphi^2 - \varphi - 1 = 0.
\]
By the quadratic formula,
\[
\varphi = \frac{1 + \sqrt{5}}{2}
\]
where the negative solution is ignored since \(\varphi\) is positive.

(b) Using a calculator we get
\[
F_{20} = \left\lfloor \frac{\varphi^{20}}{\sqrt{5}} \right\rfloor = 6765.
\]

(c) The value of \(\varphi\) is approximately 1.618 which is less than 2. Therefore, we must have \(\varphi^n \prec 2^n\) and therefore from part (b),
\[
F_n \prec 2^n.
\]
It follows that \(\lim_{n \to \infty} \frac{F_n}{2^n} = 0\). Therefore
\[
\lim_{n \to \infty} \left( \frac{\text{robots}}{\text{robots} + \text{rabits}} \right) = \lim_{n \to \infty} \frac{1}{1 + \frac{\text{rabits}}{\text{robots}}}
\]
\[
= \lim_{n \to \infty} \frac{1}{1 + \frac{F_n}{2^n}}
\]
\[
= 1.
\]

(d) From part (b) we know that \(F_n \prec \varphi^n\) and therefore \(1/F_n \prec (1/\varphi)^n\). Since \(1/\varphi < 1\), it follows that
\[
\sum_{n=1}^{\infty} \left( \frac{1}{\varphi} \right)^n < \infty
\]
and therefore by limit comparison the sum of the reciprocals of the Fibonacci numbers converges.

**Problem J**

(a) We have
\[
\log(n!) = \log(n(n-1) \ldots 2 \cdot 1) = \log n + \log(n-1) + \ldots + \log 2 + \log 1 = \sum_{k=1}^{n} \log(k).
\]
(b) See scan.

(c) The indefinite integral of \( \ln x \) is \( x \ln x - x + C \), for some constant \( C \). Therefore for any number \( a > 1 \),
\[
\int_1^a \ln x \, dx = [x \ln x - x]_1^a = a \ln a - a + 1.
\]
The inequality in (b) therefore translates to
\[
n \ln n - n + 1 \leq \sum_{k=1}^n \log(k) \leq (n + 1) \log(n + 1) - n.
\]
By part (a) the middle term is equal to \( \log(n!) \) and therefore we get the inequality
\[
n \ln n - n + 1 \leq \log(n!) \leq (n + 1) \log(n + 1) - n
\]
as desired.

(d) First note that it is convenient to write the inequality in part (c) as
\[
\ln(n^n) - n + 1 \leq \log(n!) \leq \log((n + 1)^{n+1}) - n
\]
Taking \( e \) to the power of this rearranged inequality, we get that
\[
e^{\ln n^n - n + 1} \leq n! \leq e^{\log(n+1)^{n+1} - n}.
\]
However,
\[
e^{\ln n^n - n + 1} = e^{\ln n^n} \cdot e^{-n} \cdot e = e \cdot \frac{n^n}{e^n}.
\]
Similarly,
\[
e^{\log(n+1)^{n+1} - n} = e^{\log(n+1)^{n+1}} \cdot e^{-n} = \frac{(n + 1)^{n+1}}{e^n}.
\]
Therefore the inequality in (**) now reads
\[
e \cdot \frac{n^n}{e^n} \leq n! \leq \frac{(n + 1)^{n+1}}{e^n}
\]
as desired.

(e) We have
\[
\frac{(n + 1)^{n+1}}{n^{n+1}} = \left( \frac{n + 1}{n} \right)^{n+1} = \left( 1 + \frac{1}{n} \right)^n \left( 1 + \frac{1}{n} \right).
\]
Therefore
\[
\lim_{n \to \infty} \left( \frac{(n + 1)^{n+1}}{n^{n+1}} \right) = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) = e.
\]

(f) We will use Stirling’s approximation, namely that
\[
n! \asymp n^{n+1/2}e^{-n}
\]
. We get from this that
\[
(2n)! \asymp (2n)^{2n+1/2}e^{-2n}, \quad (n!)^2 \asymp \frac{n^{2n+1}}{e^{2n}}.
\]
Hence,

\[
\frac{(2n)!}{4^n(n!)^2} \approx \left(2n\right)^{2n+1/2}e^{-2n} \cdot \left(\frac{e^{2n}}{4^n \cdot n^{2n+1}}\right)
\]

\[
= \left(2n\right)^{2n+1/2} \frac{1}{4^n \cdot n^{2n+1}}
\]

\[
= \frac{2^{2n} \cdot (2n)^{2n} \cdot \sqrt{n}}{4^n \cdot n^{2n} \cdot n}
\]

\[
= \frac{\sqrt{2}}{\sqrt{n}}.
\]

The series \(\sum_{n=0}^{\infty} \frac{2}{\sqrt{n}}\) diverges by the \(p\)-series test with \(p = 1/2\) and therefore by limit comparison the series

\[
\frac{(2n)!}{4^n(n!)^2}
\]

diverges.
Problem J (b):

Consider the figure on the left.
It is clear that
(area of boxes) \leq \text{area under } y = \ln x
for 1 \leq x \leq 5.

However, the total area of the boxes is \ln 2 + \ln 3 + \ln 4,
while the area under \( y = \ln x \) for 1 \leq x \leq 5 is given by
\[
\int_1^5 \ln x \, dx.
\]

Therefore, it follows from the picture above that
\[
\ln 2 + \ln 3 + \ln 4 \leq \int_1^5 \ln x \, dx,
\]

or more succinctly
\[
\sum_{k=1}^{n} \ln (k) \leq \int_1^5 \ln x \, dx.
\]

Noting that \( \ln(1) = 0 \).
On the other hand, if we consider instead the figure

we see that

area under curve \( \leq \) sum of area of boxes.

The term on the left is \( \int_{1}^{4} \ln x \, dx \), and the term on the right is

\[
\ln 2 + \ln 3 + \ln 4 = \sum_{k=1}^{4} \ln(k)
\]

In summary, the figure on this page gives the inequality

\[
\sum_{k=1}^{4} \ln(k) \leq \int_{1}^{4} \ln x \, dx \leq \sum_{k=1}^{5} \ln(k)
\]

Combining \( \bigcirc \) and \( \bigotimes \) gives

\[
\int_{1}^{4} \ln x \, dx \leq \sum_{k=1}^{5} \ln(k) \leq \int_{1}^{5} \ln x \, dx.
\]
Therefore in general, it is easy to see for every \( n > 1 \) that the inequality

\[
\int_1^n \ln x \, dx \leq \sum_{k=1}^{n+1} \ln k \leq \int_1^{n+1} \ln x \, dx
\]

is true.