Exercises

For exercises 1 and 2, use direct comparison to determine whether the series converges or diverges.

1. \[ \sum_{n=1}^{\infty} \frac{1}{3^n + 1} \]
   For any \( n \geq 1 \), \( 3^n + 1 > 3^n \), so \( \frac{1}{3^n + 1} < \frac{1}{3^n} \). The direct comparison test tells us that
   \[ \sum_{n=1}^{\infty} \frac{1}{3^n + 1} \leq \sum_{n=1}^{\infty} \frac{1}{3^n} \]
   and if the right-hand side of the inequality converges, so does the left-hand side.
   Now, note that \( \frac{1}{3^n} \) is a geometric series with first term \( a = \frac{1}{3} \) and common ratio \( \frac{1}{3} \). Since \( |\frac{1}{3}| < 1 \), the geometric series converges, and thus \( \sum_{n=1}^{\infty} \frac{1}{3^n} \) also converges.

   This is all the question asked for, but we can also use the comparison test to put an upper bound on the size of our series. By the formula for the sum of the geometric series, the right hand side converges to \( \frac{a}{1-r} = \frac{1/3}{2/3} = \frac{1}{2} \), so we have
   \[ \sum_{n=1}^{\infty} \frac{1}{3^n + 1} \leq \frac{1}{2} \]

2. \[ \sum_{n=1}^{\infty} \frac{n + n \cdot (\sin n)^2}{n^3 + 1} \]
   First, notice that \( (\sin n)^2 \leq 1 \) for all \( n \), so we have
   \[ n + n(\sin n)^2 \leq n + n = 2n \]
We also have $n^3 + 1 > n^3$ so

$$\frac{1}{n^3 + 1} < \frac{1}{n^3}$$

Multiplying these inequalities, we get:

$$\frac{n + n(\sin n)^2}{n^3 + 1} \leq \frac{2n}{n^3} = 2 \cdot \frac{1}{n^2}$$

Now, the direct comparison test tells us that

$$\sum_{n=1}^{\infty} \frac{n + n(\sin n)^2}{n^3 + 1} \leq \sum_{n=1}^{\infty} 2 \cdot \frac{1}{n^2}$$

and that if the right hand side converges, so does the left-hand side. Now, the right hand side is equal to $2 \sum_{n=1}^{\infty} \frac{1}{n^2}$, and this converges by the $p$-test with $p = 2$. So $\sum_{n=1}^{\infty} \frac{n + n(\sin n)^2}{n^3 + 1}$ converges.

For exercises 3-6, use the ratio test to determine whether the series converges or diverges.

3. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n + 1)/2^{n+1}}{n/2^n} \right| = \lim_{n \to \infty} \left| \frac{(n + 1)2^n}{n \cdot 2^{n+1}} \right| = \frac{1}{2} \lim_{n \to \infty} \left| \frac{n}{n + 1} \right| = \frac{1}{2}$$

This is less than 1, so $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

4. $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n + 1)!(n + 2)!/(2(n + 1))!}{n!(n + 1)!/(2n)!} \right| = \lim_{n \to \infty} \left| \frac{(n + 1)!(n + 2)!}{n!(n + 1)!}(2n)! \right| = \lim_{n \to \infty} \left| \frac{(n + 2)!}{(2n)!} \cdot \frac{n + 1}{2n + 2} \right| = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

This is less than 1, so $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$ converges.
5. \[ \sum_{n=1}^{\infty} \frac{1}{r^n n!}, \quad r > 0, \]

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{1/(r^{n+1}(n+1)!)}{1/(r^n n!)} \right| \\
= \lim_{n \to \infty} \left| \frac{r^n n!}{r^{n+1}(n+1)!} \right| \\
= \lim_{n \to \infty} \left| \frac{1}{r(n+1)} \right| \\
= \frac{1}{|r|} \lim_{n \to \infty} \frac{1}{n+1} \\
= 0
\]

This is less than 1, so \( \sum_{n=1}^{\infty} \frac{1}{r^n n!} \) converges.

6. \[ \sum_{n=0}^{\infty} \frac{2^n}{n^3 + 1} \]

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{2^{n+1}/((n+1)^3 + 1)}{2^n/(n^3 + 1)} \right| \\
= \lim_{n \to \infty} \left| \frac{2^{n+1}}{2^n} \cdot \frac{n^3 + 1}{(n+1)^3 + 1} \right| \\
= 2 \cdot \lim_{n \to \infty} \left| \frac{n^3 + 1}{(n+1)^3 + 1} \right| \\
= 2 \cdot 1 \\
= 2
\]

We used the law of the dominant term for the last limit: \( (n+1)^3 = n^3 + 3n^2 + 3n + 1 \). This is greater than 1, so \( \sum_{n=1}^{\infty} \frac{2^n}{n^3 + 1} \) diverges.

For exercises 7-10, use limit comparison to determine whether the series converges or diverges.

7. \[ \sum_{n=1}^{\infty} \frac{5n + 1}{3n^2} \]

Since the term \( a_n \) is a linear function of \( n \) divided by a quadratic function of \( n \), we should compare it to \( \frac{n}{n^2} = \frac{1}{n} \). Indeed, we have:

\[
\lim_{n \to \infty} \left| \frac{(5n + 1)/(3n^2)}{1/n} \right| = \lim_{n \to \infty} \left| \frac{5n^2 + n}{3n^2} \right| = \frac{5}{3}
\]

Here, we used the law of the dominant term. Thus,

\[ \frac{5n + 1}{n^3} \sim \frac{1}{n} \]

This says that \( \sum_{n=1}^{\infty} \frac{5n + 1}{n^3} \) converges if and only if \( \sum_{n=1}^{\infty} \frac{1}{n} \) converges. But this is the harmonic series, which diverges by the \( p \)-test with \( p = 1 \). Thus, \( \sum_{n=1}^{\infty} \frac{5n + 1}{3n^2} \) diverges.
8. \( \sum_{n=1}^{\infty} \left( 1 - \cos \frac{1}{n} \right) \) \hspace{1cm} \text{Hint: compare with} \sum \frac{1}{n^2} \text{ We follow the hint, and compare } a_n = 1 - \cos \frac{1}{n} \text{ with } \frac{1}{n^2}. \text{ We have:} \\
\lim_{n \to \infty} \left| \frac{1 - \cos \frac{1}{n}}{1/n^2} \right| = \lim_{n \to \infty} \left| \frac{1 - \cos \frac{1}{x}}{1/x^2} \right| \to 0 \\
This is an indeterminate form, so we may apply L’Hôpital’s Rule. This tells us: \\
\lim_{x \to \infty} \left| \frac{\frac{\sin \frac{1}{x}}{1/x^2}}{1/x} \right| = \lim_{x \to \infty} \left| \frac{\sin \frac{1}{x}}{1/x} \right| \to 0 \\
Thus, we may apply L’Hôpital’s Rule again: \\
\lim_{x \to \infty} \left| \sin \frac{1}{x} \right| = \lim_{x \to \infty} \left| \cos \frac{1}{x} \right| = 1 \\
Thus, we see that \\
\left( 1 - \cos \frac{1}{n} \right) \asymp \frac{1}{n^2} \\
This tells us that \( \sum_{n=1}^{\infty} \left( 1 - \cos \frac{1}{n} \right) \) converges if and only if \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges. But this converges by the \( p \)-test with \( p = 2 \). Thus, the series \( \sum_{n=1}^{\infty} \left( 1 - \cos \frac{1}{n} \right) \) converges.

9. \( \sum_{n=10}^{\infty} \frac{4 \sin n + n}{n^2} \) \hspace{1cm} \text{The numerator is a bounded term } 4 \sin n \text{ plus a linear term } n, \text{ so the linear term is the dominant term. Thus, we expect } a_n = \frac{4 \sin n + n}{n^2} \text{ to be asymptotic to } \frac{n}{n^2} = \frac{1}{n}. \text{ Let’s show this:} \\
\lim_{n \to \infty} \left| \frac{(4 \sin n + n)/n^2}{1/n} \right| = \lim_{n \to \infty} \left| \frac{4 \sin n + n}{n} \right| = \lim_{n \to \infty} \left| \frac{4 \sin n}{n} \right| + \lim_{n \to \infty} \left| \frac{n}{n} \right| = \lim_{n \to \infty} \left| \frac{4 \sin x}{x} \right| + 1 = 0 + 1 = 1 \\
We can see that \( \lim_{x \to \infty} \left| \frac{4 \sin x}{x} \right| = 0 \) because \( 0 \leq |4 \sin x| \leq 1 \) and thus \( 0 \leq \left| \frac{4 \sin x}{x} \right| \leq \frac{1}{x} \). Both the left and right-hand side converge to 0 as \( x \to \infty \), so the middle does as well by the squeeze theorem. Thus: \\
\frac{4 \sin n + n}{n^2} \asymp \frac{1}{n} \)
Thus, $\sum_{n=10}^{\infty} \frac{4 \sin n + n}{n^2}$ converges if and only if $\sum_{n=10}^{\infty} \frac{1}{n}$ converges. This diverges by the $p$-test with $p = 1$ (note that changing the lower index in the sum does not effect convergence), so $\sum_{n=10}^{\infty} \frac{4 \sin n + n}{n^2}$ diverges.

10. $\sum_{n=5}^{\infty} \frac{2^n}{3^n - 2}$ The law of the dominant term tells us that we should be able to ignore the term “$-2$” in the denominator, and $a_n = \frac{2^n}{3^n - 2}$ will be asymptotic to $\frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$. Let’s see that this is the case:

$$\lim_{n \to \infty} \left| \frac{2^n / (3^n - 2)}{2^n / 3^n} \right| = \lim_{n \to \infty} \left| \frac{2^n : 3^n}{2^n (3^n - 2)} \right|$$
$$= \lim_{n \to \infty} \left| \frac{3^n}{3^n - 2} \right|$$
$$= 1$$

**Problem I: The Fibonacci sequence**

The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

a. Though the Fibonacci sequence is not literally geometric, the limit

$$\varphi = \lim_{n \to \infty} \left( \frac{F_{n+1}}{F_n} \right)$$

exists and is positive. Find this limit.

*Hint 1: Remember that $F_{n+1}$ is the sum of the two previous terms.*

*Hint 2: If $\frac{F_{n+1}}{F_n} \to \varphi$ as $n \to \infty$, then what is the limit of $\frac{F_{n-1}}{F_n}$ as $n \to \infty$?*

*Hint 3: Perhaps unexpectedly, you’re going to need the quadratic formula!*

Let $\varphi = \lim_{n \to \infty} \left( \frac{F_{n+1}}{F_n} \right)$. Then we have (for $n \geq 1$):

$$\varphi = \lim_{n \to \infty} \left( \frac{F_{n+1}}{F_n} \right)$$
$$= \lim_{n \to \infty} \left( \frac{F_n + F_{n-1}}{F_n} \right)$$
$$= \lim_{n \to \infty} \left( 1 + \frac{F_{n-1}}{F_n} \right)$$
$$= 1 + \frac{1}{\varphi}$$
Note that \( \lim_{n \to \infty} \left( \frac{F_{n-1}}{F_n} \right) = \lim_{n \to \infty} \left( \frac{F_n}{F_{n+1}} \right) \), since we are just shifting the indexing of the sequence by 1. Thus, we have the equation \( \varphi = 1 + \frac{1}{\varphi} \). Now we are given that \( \varphi > 0 \), so in particular \( \varphi \neq 0 \), so this makes sense. We can multiply everything by \( \varphi \) and rearrange to get

\[
\varphi^2 - \varphi - 1 = 0
\]

By plugging this into the quadratic formula, we get:

\[
\varphi = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2}
\]

Now, we are also given that \( \varphi \) is positive, so the right answer is

\[
\varphi = \frac{1 + \sqrt{5}}{2}
\]

b. By (a), \( F_{n+1} \approx \varphi \cdot F_n \) for large indices \( n \), so \( F_n \simeq \varphi^n \) as \( n \to \infty \). More precisely, we have the exact, closed formula

\[
F_n = \left[ \frac{\varphi^n}{\sqrt{5}} \right]
\]

where \([y]\) means “\( y \) rounded to the nearest integer.” Use this (and a calculator with an exponentiation function) to find \( F_{20} \), the 20th Fibonacci number.

The above formula is exact, so we just need to plug in \( n = 20 \). This tells us:

\[
F_{20} = \left[ \frac{\varphi^{20}}{\sqrt{5}} \right] = \left[ 6765.000295639 \right] = 6765
\]

c. Though he was not the first to study this sequence, it is named after the 12th century Italian mathematician Leonardo “Fibonacci” of Pisa. He discovered the sequence while imagining an idealized population growth model for rabbits. (Look it up, if you’re curious.)

Suppose that PLANET R is home to robots and rabbits. If the number of robots after \( n \) weeks is \( 2^n \) and the number of rabbits after \( n \) weeks is \( F_n \), then who will eventually dominate the planet? That is, one of

\[
\lim_{n \to \infty} \left( \frac{\text{robots}}{\text{robots} + \text{rabbits}} \right) \quad \text{or} \quad \lim_{n \to \infty} \left( \frac{\text{rabbits}}{\text{robots} + \text{rabbits}} \right)
\]

will be equal to 1. Which, and why?

Let \( R_n \) be the number of robots after \( n \) weeks, so we have \( R_n = 2^n \). Since \( F_n \simeq \varphi^n \) and \( \varphi \approx 1.618 < 2 \), we should expect the number of robots to grow faster. Thus, we expect the first limit to be 1. Let’s check this:

\[
\lim_{n \to \infty} \left( \frac{R_n}{R_n + F_n} \right) = \lim_{n \to \infty} \left( \frac{2^n}{2^n + F_n} \right) = \frac{1}{1 + \lim_{n \to \infty} 2^{-n} F_n}
\]
Since $F_n \approx \varphi^n$, $2^{-n} F_n \approx (\varphi/2)^n$. But $|\varphi/2| < 1$, so $\lim_{n \to \infty} (\varphi/2)^n = 0$. Thus, $\lim_{n \to \infty} 2^{-n} F_n = 0$ as well. This tells us:

$$\lim_{n \to \infty} \left( \frac{R_n}{R_n + F_n} \right) = 1$$

Therefore, robots will eventually dominate.

d. Determine whether $\sum_{n=1}^{\infty} \frac{1}{F_n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \cdots$ converges or diverges.

We will use the limit comparison test, together with the fact (established in part a.) that $F_n \approx \varphi^n$ as $n \to \infty$. This tells us that $\frac{1}{F_n} \approx \frac{1}{\varphi^n}$, so $\sum_{n=1}^{\infty} \frac{1}{F_n}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{\varphi^n}$ converges. Now, this latter series is geometric with first term $a = \frac{1}{\varphi}$ and common ratio $r = \frac{1}{\varphi}$. Since $\varphi \approx 1.618 > 1$, $|1/\varphi| < 1$, so the geometric series converges. Thus, $\sum_{n=1}^{\infty} \frac{1}{F_n}$ converges.

**Problem J: Stirling’s Approximation**

In many fields (computer science especially) it is important to have a handle on how quickly the sequence of factorials $n!$ grows. In class, we learned that

any exponential growth $r^n \prec n! \prec n^n$

However, we can find a much more precise estimate by comparing sums and integrals.

a. Let $n \geq 1$. Explain why $\ln(n!) = \sum_{k=1}^{n} \ln(k)$.

Remember the product-sum formula for logs: $\ln(ab) = \ln(a) \ln(b)$. Since $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$, we can apply this repeatedly to get:

$$\ln(n!) = \ln(n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1) = \ln(n) + \ln(n-1) + \cdots + \ln(2) + \ln(1) = \sum_{k=1}^{n} \ln(k)$$

If we wanted to avoid $(\cdots)$ expressions and to be a bit more rigorous, we could prove this statement by induction: $\ln(1!) = \ln(1) = 0 = \sum_{k=1}^{1} \ln(k)$, so the base case $n = 1$ holds. Now assume that $\ln(n!) = \sum_{k=1}^{n} \ln(k)$ for some $n$. We have

$$\ln((n + 1)!) = \ln((n + 1)n!) = \ln(n + 1) + \ln(n!) = \ln(n + 1) + \sum_{k=1}^{n} \ln(k) = \sum_{k=1}^{n+1} \ln(k)$$

b. Comparing sums and integrals (as in our explanation for the integral test!—except it’s important that $\ln x$ is increasing and not decreasing), we have the inequality

$$\int_{1}^{n} \ln x \, dx \leq \sum_{k=1}^{n} \ln(k) \leq \int_{1}^{n+1} \ln x \, dx$$
Justify this inequality by drawing the relevant picture for \( n = 4 \), noting that the sum \( \sum_{k=1}^{4} \ln(k) \) is equal to the 4th left Riemann sum for \( \int_{1}^{5} \ln x \, dx \) and also equal to the 3rd right Riemann sum for \( \int_{1}^{4} \ln x \, dx \).

First, we draw the left Riemann sum for \( \int_{1}^{5} \ln x \, dx \):

![Left Riemann Sum](image)

The intervals are \([1, 2], [2, 3], [3, 4], \) and \([4, 5] \), so the left endpoints are at \( x = 1, 2, 3, 4 \). Thus, the left Riemann sum for \( \int_{1}^{5} \ln x \, dx \) is

\[
\ln(1) + \ln(2) + \ln(3) + \ln(4) = \sum_{k=1}^{4} \ln(k)
\]

As we can see from the picture, the fact that the function \( \ln x \) is increasing tells us that the left Riemann sum is smaller than the integral, so we get the inequality:

\[
\sum_{k=1}^{4} \ln(k) \leq \int_{1}^{5} \ln x \, dx
\]

Next, we draw the right Riemann sum for \( \int_{1}^{4} \ln x \, dx \):

![Right Riemann Sum](image)

The intervals are \([1, 2], [2, 3], \) and \([3, 4] \), so the right endpoints are at \( x = 2, 3, 4 \). Thus, the right Riemann sum for \( \int_{1}^{4} \ln x \, dx \) is:

\[
\ln(2) + \ln(3) + \ln(4) = \sum_{k=2}^{4} \ln k = \ln(1) + \sum_{k=2}^{4} \ln k = \sum_{k=1}^{4} \ln k
\]

Here, we used the fact that \( \ln(1) = 0 \) to get the second equality.
c. Combine (a.) and the integral formula for \( \int \ln x \, dx \) (in your textbook) to conclude that

\[
n \ln n - n + 1 \leq \ln(n!) \leq (n + 1) \ln(n + 1) - n
\]

By part (a.), \( \ln(n!) = \sum_{k=1}^{n} \ln(k) \). Plugging this into part (b.) tells us that:

\[
\left( x \ln x - x \right) \bigg|_{1}^{n} \leq \ln(n!) \leq \left( x \ln x - x \right) \bigg|_{1}^{n+1}
\]

\[
n \ln n - 1 \leq \ln(n!) \leq ((n + 1) \ln(n + 1) - (n + 1)) - (1 \ln 1 - 1)
\]

We used the identity \( \int \ln x \, dx = x \ln x - x + C \).

d. Raising \( e \) to the inequality from (c.), and simplifying carefully, show that

\[
e \cdot n^\frac{n}{e^n} \leq n! \leq \frac{(n + 1)^{n+1}}{e^n}
\]

If \( a \leq b \leq c \), then \( e^a \leq e^b \leq e^c \), since \( e^x \) is monotonically increasing. Now we can raise \( e \) to both sides of the inequality in part (c.):

\[
e^{n \ln n - n + 1} \leq e^{\ln(n!)} \leq e^{(n + 1) \ln(n + 1) - n}
\]

\[
e^n \ln n \cdot e^{-n} \cdot e \leq n! \leq e^{(n + 1) \ln(n + 1)} \cdot e^{-n}
\]

\[
e \cdot \frac{n^a}{e^n} \leq n! \leq \frac{(n + 1)^{n+1}}{e^n}
\]

Here, we used the fact that \( e^{a \ln b} = b^a \). This follows from the fact that \( a \ln b = \ln(b^a) \) and \( e^{\ln b^a} = b^a \) or that \( e^{xy} = (e^x)^y = (e^y)^x \) so \( e^{a \ln b} = (e^{\ln b})^a = b^a \).

e. Show that \( \lim_{n \to \infty} \left( \frac{(n + 1)^{n+1}}{n^{n+1}} \right) = e \)

**Hint:** \( 1^\infty \) is an indeterminate form. You will need to use the special limit \( \lim_{n \to \infty} (1 + \frac{1}{n})^n = e \).

We can simplify the expression in the limit so that we can apply the special limit \( \lim_{n \to \infty} (1 + \frac{1}{n})^n = e \):

\[
\lim_{n \to \infty} \left( \frac{(n + 1)^{n+1}}{n^{n+1}} \right) = \lim_{n \to \infty} \left( \frac{n + 1}{n} \right)^{n+1}
\]

\[
= \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{n+1}
\]

\[
= \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) \cdot \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)
\]

\[
= e \cdot (1 + 0) = e
\]
So for large \( n \), the upper bound on \( n! \) in (d) is \( \approx e \cdot \frac{n^{n+1}}{e^n} \). Replacing our upper bound with this approximation, we get \((e \cdot \frac{n^n}{e^n}) \leq n! \leq n \left( e \cdot \frac{n^n}{e^n} \right)\) which is true for all \( n \geq 1 \) (note that the terms in the parentheses on left and right are the same).

So what’s the correct asymptotic for \( n! \)? It turns out the answer is in some sense “half-way” between the two sides of the inequality above:

\[
n! \asymp n^{1/2} \left( \frac{n^n}{e^n} \right) = n^{n+1/2}e^{-n}
\]

This approximation (and more precise versions of it) are called Stirling’s approximation for the factorial. There are a couple series in this class whose convergence can only be determined using Stirling’s approximation. For example:

f. Determine whether the series \( \sum_{n=0}^{\infty} \frac{(2n)!}{4^n \cdot (n!)^2} \) converges or diverges.

*Hint: The Ratio Test will be inconclusive. *Carefully* try limit comparison, using Stirling’s approximation.

First let’s see that the Ratio Test doesn’t work:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(2(n + 1))!/((4^{n+1} \cdot ((n+1)!)^2)}{(2n)!/(4^n \cdot (n!)^2)} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{(2n + 2)!}{(2n)!} \right| \cdot \left| \frac{4^n(n!)^2}{4^{n+1}((n+1)!)^2} \right|
\]

\[
= \lim_{n \to \infty} \frac{1}{4} \left| \frac{(2n + 2)(2n + 1)}{(n + 1)^2} \right|
\]

\[
= 1
\]

We used the law of the dominant term in the last step to write

\[
\lim_{n \to \infty} \left| \frac{(2n + 2)(2n + 1)}{(n + 1)^2} \right| = \lim_{n \to \infty} \left| \frac{4n^2}{n^2} \right| = 4
\]

We see that the Ratio Test is indeed conclusive. Now, we will use Stirling’s approximation to find a sequence which is asymptotic to \( a_n = \frac{(2n)!}{4^n(n!)^2} \). Since \( n! \asymp n^{n+1/2}e^{-n} \), we have

\[
(n!)^2 \asymp (n^{n+1/2} \cdot e^{-n})^2 = n^{2n+1} \cdot e^{-2n}
\]

Applying Stirling’s approximation again, we have:

\[
(2n)! \asymp \left( (2n)^{(2n)+1/2} \cdot e^{-2n} \right) = 2^{2n} \cdot 2^{1/2} \cdot n^{2n+1/2} \cdot e^{-2n} = \sqrt{2} \cdot 4^n \cdot n^{2n+1/2} \cdot e^{-2n}
\]
Thus, we have:

\[
\frac{(2n)!}{4^n(n!)^2} \leq \sqrt{2} \cdot \frac{4^n \cdot n^{2n+1/2} \cdot e^{-2n}}{4^n \cdot n^{2n+1} \cdot e^{-2n}} = \sqrt{2} \cdot \frac{n^{2n} \cdot n^{1/2}}{n^{2n} \cdot n} = \sqrt{2} \cdot \frac{1}{\sqrt{n}}
\]

This tells us that \( \sum_{n=1}^{\infty} \frac{(2n)!}{4^n(n!)^2} \) converges if and only if \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) converges. But this latter series is a constant multiple of the \( p \)-series with \( p = 1/2 \), so it diverges by the \( p \)-test. Thus, \( \sum_{n=1}^{\infty} \frac{(2n)!}{4^n(n!)^2} \) \textbf{diverges.}