Math 21: Homework 4 Solutions

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1. Conceptual Questions

Problem 1.

1. We’re allowed to use \( \sum (A \pm B) = \sum A \pm \sum B \) whenever the infinite series \( \sum A \) and \( \sum B \) both converge.

2. Step (iii) is incorrect because we’re not allowed to do Step (i), and also we’re not allowed to subtract \( \infty \) from \( \infty \). We need to know how “big” these infinities are relative to each other for example.

3. Looking at each partial sum of the infinite series \( \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) \), we see the partial sums are finite sums of positive numbers, since \( \frac{1}{k} > \frac{1}{k+1} \). Therefore, the partial sums are monotone increasing, and each partial sum is positive. If this limit exists, i.e., if the infinite series converges, then the limit must be positive. If this limit does not exist, the partial sums must go to infinity, i.e., \( \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = +\infty \).

4. We rewrite the infinite series as follows by simplifying the terms in the series using common denominators:

\[
\sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=1}^{\infty} \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} \tag{1.1}
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \tag{1.2}
\]

But \( \frac{1}{k(k+1)} = \frac{1}{k^2} \) is asymptotic to \( \frac{1}{k^2} \). By the \( p \)-test, the infinite series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) converges. Thus, since every term in the infinite series \( \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) \) is positive, we can use the limit comparison test to deduce that this infinite series converges.

5. We have the identity

\[
\sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} \tag{1.3}
\]

because for each \( k \) between 1 and \( n \), the term \( -\frac{1}{k+1} \) will cancel with the \( +\frac{1}{k+1} \) coming from the next value of \( k \). Thus, we are left with the term \( \frac{1}{k} \) for \( k = 1 \) and the term \( \frac{1}{k+1} \) for \( k = n \). This gives \( 1 - \frac{1}{n+1} \). Thus,

\[
\sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1. \tag{1.4}
\]

Problem 2. Give examples of:

1. Let \( a_n = \frac{1}{n^2} \). Then \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges by the \( p \)-test. Moreover, we also know \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \).

2. Let \( a_n = \frac{1}{n} \). Then we know \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} \) diverges also by the \( p \)-test. Moreover, we still have \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

3. Let \( a_n = \frac{1}{n^2} \). Then as above we know \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges. But \( \sum_{n=1}^{\infty} \sqrt{a_n} = \sum_{n=1}^{\infty} \frac{1}{n} \) diverges!

Problem 3. For each of the test/series pairs below, explain why that test cannot be applied to that series.

1. We cannot apply the integral test for \( \sum_{k=1}^{\infty} \frac{\sin(k)^2}{k^2} \) because the function \( f(x) = \frac{\sin(x)^2}{x^2} \) is not decreasing.

2. We cannot apply the integral test for \( \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2} \) because the function \( f(x) = \frac{(-1)^x}{\ln(x)} \) is not always positive.

3. We cannot apply the integral test for \( \sum_{k=6}^{\infty} \frac{\sec(k-1)}{k} \) because the function \( f(x) = \sec(x) - 1 \) is not always continuous (it has vertical asymptotes).
(4) We cannot apply the direct comparison test for \( \sum_{k=1}^{\infty} \frac{\cos k}{k} \) because \( \frac{\cos k}{k} \) is not always \( > 0 \).

### 2. Routine Problems

**Problem 4.**

1. We use the divergence test. Namely, \( \lim_{k \to \infty} k = \infty \), which is clearly nonzero. This implies that the series \( \sum_{k=1}^{\infty} k \) diverges.

2. We use the divergence test. Namely, \( \lim_{k \to \infty} (-1)^k \cdot k \) does not exist. This implies that the series \( \sum_{k=1}^{\infty} (-1)^k \cdot k \) diverges.

3. For any \( k \), we see that
   \[
   (-1)^k + (-1)^{k+1} = 0.
   \]
   This is because \( (-1)^{k+1} = -(-1)^k \). Therefore,
   \[
   \sum_{k=1}^{\infty} [(-1)^k + (-1)^{k+1}] \cdot k = \sum_{k=1}^{\infty} 0 \cdot k = \sum_{k=1}^{\infty} 0.
   \]
   This implies \( \sum_{k=1}^{\infty} [(-1)^k + (-1)^{k+1}] \cdot k = 0 \), so it converges by direct evaluation.

**Problem 5.**

1. First note, for any \( k \geq 4 \), that \( \frac{1}{x(\ln x)^2} \geq 0 \). Moreover, we know \( \frac{1}{x(\ln x)^2} < \frac{1}{x^2} \), and \( \sum_{k=4}^{\infty} \frac{1}{k^2} \) converges by the p-test. Therefore, \( \sum_{k=4}^{\infty} \frac{1}{k(\ln k)^2} \) converges by the limit comparison test.

2. First note that the function \( f(x) = \frac{1}{x(\ln x)^2} \) is positive, decreasing, and continuous for \( x \geq 4 \). Thus, to determine whether the series \( \sum_{k=4}^{\infty} \frac{1}{k(\ln k)^2} \) converges or diverges, by the integral test it suffices to examine
   \[
   \int_{4}^{\infty} \frac{1}{x(\ln x)^2} \, dx.
   \]
   We compute this last integral directly by \( u \)-substitution, letting \( u = \ln x \). Thus,
   \[
   \int_{4}^{\infty} \frac{1}{x(\ln x)^2} \, dx = \lim_{b \to \infty} \int_{4}^{b} \frac{1}{x(\ln x)^2} \, dx
   = \lim_{b \to \infty} \int_{\ln 4}^{\ln b} \frac{1}{u^2} \, du
   = \lim_{b \to \infty} \left[ -\frac{1}{u} \right]_{\ln 4}^{\ln b}
   = \lim_{b \to \infty} \left[ -\frac{1}{\ln b} + \frac{1}{\ln 4} \right]
   = \frac{1}{\ln 4}.
   \]
   Thus, this improper integral converges, and by the integral test this means the series \( \sum_{k=4}^{\infty} \frac{1}{k(\ln k)^2} \) converges.

3. First note \( (\ln k)^2 \geq 0 \) for any \( k \geq 4 \). Now note that \( (\ln k)^2 < k^{1/2} \), so that \( (\ln k)^2 < \frac{k^{1/2}}{k^2} = k^{-3/2} \). But \( \sum_{k=4}^{\infty} k^{-3/2} \) converges by the p-test. By the limit comparison test, this tells us \( \sum_{k=4}^{\infty} (\ln k)^2 \) converges.

**Problem 6.**

1. First note, for any \( n \),
   \[
   \frac{(-1)^n(-2)^n(-3)^n}{(-4)^n(-5)^n} = \left[ \frac{(-1)(-2)(-3)}{(-4)(-5)} \right]^n = \left( \frac{-3}{10} \right)^n.
   \]
   Thus,
   \[
   \sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n(-3)^n}{(-4)^n(-5)^n} = \sum_{n=0}^{\infty} \left( \frac{-3}{10} \right)^n.
   \]
This means the series is a geometric series with rate \( r = -\frac{3}{10} \). Because \(|r| < 1\), the geometric series converges.

(2) First note that \( \left( \frac{1}{4} \right)^n \geq 0 \) for ever \( n \geq 2 \). Moreover, note that \( \frac{1}{n} \leq \frac{1}{2} \) for every \( n \geq 2 \). This implies that \( \left( \frac{1}{4} \right)^n \leq \left( \frac{1}{2} \right)^n \) for every \( n \geq 2 \) as well. Moreover, the series \( \sum_{n=2}^{\infty} \left( \frac{1}{2} \right)^n \) converges since it is a geometric series with rate \( r = \frac{1}{2} \).

Thus, by the direct comparison test, we deduce the series \( \sum_{n=2}^{\infty} \left( \frac{1}{n} \right)^n \) converges as well.

(3) Note that

\[
 n^{(-1)^p} = \begin{cases} 
 \frac{1}{n} & \text{if } n \text{ odd} \\
 n & \text{if } n \text{ even}
\end{cases}
\]  

Thus, \( \lim_{n \to \infty} n^{(-1)^p} \) does not exist, which means the series \( \sum_{n=2}^{\infty} n^{(-1)^p} \) diverges by the divergence test.

**Problem 7.**

(1) We first write, upon isolating the term for \( n = 0 \),

\[
\sum_{n=0}^{\infty} \frac{1}{4n^2+1} = 1 + \sum_{n=1}^{\infty} \frac{1}{4n^2+1}.
\]

Because the infinite series on the left and the infinite series on the right differ by exactly 1, to see whether or not the infinite series on the left converges it suffices to examine whether or not the infinite series on the right converges. To this end, we note \( \frac{1}{4n^2+1} \geq \frac{1}{n^2} \). Moreover, \( \frac{1}{4n^2+1} \geq 0 \) for every \( n \geq 1 \). Since the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges by the \( p \)-test, we know \( \sum_{n=1}^{\infty} \frac{1}{4n^2+1} \) converges as well, and thus so does \( \sum_{n=0}^{\infty} \frac{1}{4n^2+1} \).

(2) We first note \( \sqrt{n} + 1 \geq \sqrt{n} \), so that \( (\sqrt{n} + 1)^6 \geq \sqrt{n}^6 = n^3 \). Thus, we see \( \frac{n^3}{(\sqrt{n}+1)^6} \leq \frac{n}{n^3} = \frac{1}{n} \). In particular, \( \lim_{n \to \infty} \frac{n^3}{(\sqrt{n}+1)^6} \) does not exist. By the divergence test, we see the series \( \sum_{n=1}^{\infty} \frac{n^3}{(\sqrt{n}+1)^6} \) diverges.

(3) First, note \( 3n+1 \leq n \) and that \( 4n+1 \geq n \). Thus, \( (3n+1)^2 \leq n^3 \) and \( (4n+1)^4 \leq n^4 \). This implies \( \frac{(3n+1)^2}{n^3} \leq \frac{1}{n} \).

By the \( p \)-test, the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges. Moreover, \( \frac{(3n+1)^2}{(4n+1)^4} \geq 0 \) for every \( n \geq 1 \). Thus, by the limit comparison test, we know \( \sum_{n=1}^{\infty} \frac{(3n+1)^2}{(4n+1)^4} \) diverges.

**Problem 8.**

(1) We first isolate the term for \( n = 0 \) and write

\[
\sum_{n=0}^{\infty} \left( \frac{3n+1}{4n+1} \right)^n = 1 + \sum_{n=1}^{\infty} \left( \frac{3n+1}{4n+1} \right)^n.
\]

Because the infinite series on the left and the infinite series on the right differ by exactly 1, to determine whether or not the infinite series on the left converges it suffices to determine whether or not the infinite series on the right converges. To this end, we note that for every \( n \geq 1 \),

\[
\frac{3n+1}{4n+1} \leq \frac{4}{5}.
\]

This can be seen by noting for \( n = 1 \), we have \( \frac{3n+1}{4n+1} = \frac{4}{5} \), and that the derivative of \( f(x) = \frac{3x+1}{4x+1} \) is negative for \( x \geq 1 \). Moreover, we know \( \sum_{n=1}^{\infty} \left( \frac{4}{5} \right)^n \) converges since it is a geometric series with rate \( r = \frac{4}{5} \) absolute value less than 1. Moreover, for any \( n \geq 1 \), we know \( \left( \frac{3n+1}{4n+1} \right)^n \geq 0 \). Thus, by the direct comparison test, we know \( \sum_{n=1}^{\infty} \left( \frac{3n+1}{4n+1} \right)^n \) converges, and thus so does \( \sum_{n=0}^{\infty} \left( \frac{3n+1}{4n+1} \right)^n \).

(2) For every \( n \geq 1 \), we know \( \frac{\sin n}{n^2} \geq 0 \). Moreover, for any \( n \), we also know \( |\sin n| \leq 1 \). Thus, \( \frac{|\sin n|}{n^2} \leq \frac{1}{n^2} \). By the \( p \)-test, we know \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges. Thus, by the direct comparison test, we know \( \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \) converges as well.

(3) First, for any \( n \geq 2 \), we know \( \frac{\ln n}{\sqrt{n}} \geq 0 \). Moreover, we know \( \ln n \geq 1 \), so \( \frac{\ln n}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} \). By the \( p \)-test, we know \( \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \) diverges. Thus, by the limit comparison test, we know \( \sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}} \) also diverges.
3. In-Depth Problems

Problem 9.

(1) Let’s call each term in parentheses a group, purely for convenience.

(a) The first group is just $\frac{1}{2}$, which is clearly $\geq \frac{1}{2}$.

(b) The second group is a collection of 2 terms, each of which is bigger than $\frac{1}{4}$. Thus, the sum of terms in the second group is $\geq 2 \cdot \frac{1}{4} = \frac{1}{2}$.

(c) The third group is a collection of 4 terms, each of which is at least $\frac{1}{8}$. Thus, the sum of terms in the third group is $\geq 4 \cdot \frac{1}{8} = \frac{1}{2}$.

(d) The fourth group is a collection of 8 terms, each of which is at least $\frac{1}{16}$. Thus, the sum of terms in the fourth group is $\geq 8 \cdot \frac{1}{16} = \frac{1}{2}$.

(2) Notice that in the sum $\sum_{n=1}^{2^m} \frac{1}{n}$, outside of the term for $n = 1$, there are $m$ groups, where the $k$-th group is of the form $\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \ldots + \frac{1}{2^{k+1}}$. This $k$-th group has $2^k$ terms, and each term is at least $\frac{1}{2^{k+1}}$. Thus, the sum of the terms in the $k$-th group is $\geq 2^k \cdot \frac{1}{2^{k+1}} = \frac{1}{2}$. This implies that the groups in the sum $\sum_{n=1}^{2^m} \frac{1}{n}$ give a sum of at least $\frac{1}{2}$ each, and because there are $m$ groups, the total sum of the $m$ groups is $m/2$. Outside of the $m$ groups, there is still the extra term for $n = 1$. Thus

$$\sum_{n=1}^{2^m} \frac{1}{n} \geq \frac{1}{m} + \frac{m}{2} \text{ terms}.$$ (3.1)

(3) For any $m$, we know

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \sum_{n=1}^{m} \frac{1}{n},$$ (3.2)

since the infinite series on the left-hand side is just a sum over more terms, and each term in the sum is positive. By part (2), this tells us

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq 1 + \frac{m}{2},$$ (3.3)

for any $m$. Thus, we may take a limit as $m \rightarrow \infty$ to see

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \lim_{m \rightarrow \infty} \left(1 + \frac{m}{2}\right) = +\infty.$$ (3.4)

This implies that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Problem 10.

(1) In the problem statement, we’re given

$$n \cdot I_{2n+1}I_{2n+2} \leq \left(\int_0^{\sqrt{n}} e^{-t^2} \, dt\right)^2 \leq n \cdot I_{2n-3}I_{2n-2}.$$ (3.5)

We’re also given, for any $k$,

$$I_kI_{k+1} = \frac{1}{k+1} \cdot \frac{1}{2}.$$ (3.6)

Thus, plugging in $k = 2n + 1$ and $k = 2n - 3$, we see

$$n \cdot I_{2n+1}I_{2n+2} = n \cdot \frac{1}{2n+2} \cdot \frac{1}{2},$$ (3.7)

$$n \cdot I_{2n-3}I_{2n-2} = n \cdot \frac{1}{2n-2} \cdot \frac{1}{2}.$$ (3.8)
Combining these inequalities with (3.2), we see
\[
\frac{n}{2n+2} \cdot \frac{\pi}{2} \leq \left(\int_0^{\sqrt{n}} e^{-t^2} \, dt\right)^2 \leq \frac{n}{2n-2} \cdot \frac{\pi}{2}.
\] (3.9)

Taking square roots, we see
\[
\sqrt{\frac{n}{2n+2} \cdot \frac{\pi}{2}} \leq \int_0^{\sqrt{n}} e^{-t^2} \, dt \leq \sqrt{\frac{n}{2n-2} \cdot \frac{\pi}{2}}.
\] (3.10)

(2) We first note
\[
\lim_{n \to \infty} \frac{n}{2n+2} = \frac{1}{2}, \quad \lim_{n \to \infty} \frac{n}{2n-2} = \frac{1}{2}.
\] (3.11)

Thus,
\[
\lim_{n \to \infty} \sqrt{\frac{n}{2n+2} \cdot \frac{\pi}{2}} = \lim_{n \to \infty} \left(\sqrt{\frac{n}{2n+2} \cdot \frac{\pi}{2}}\right) = \frac{\sqrt{\pi}}{2} \lim_{n \to \infty} \sqrt{\frac{n}{2n+2}} = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{\pi}}{2}.
\] (3.12)

Similarly,
\[
\lim_{n \to \infty} \sqrt{\frac{n}{2n-2} \cdot \frac{\pi}{2}} = \lim_{n \to \infty} \left(\sqrt{\frac{n}{2n-2} \cdot \frac{\pi}{2}}\right) = \frac{\sqrt{\pi}}{2} \lim_{n \to \infty} \sqrt{\frac{n}{2n-2}} = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{\pi}}{2}.
\] (3.13)

Thus, by part (1) and the squeeze theorem, we know
\[
\int_0^{\infty} e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2}
\] (3.20)
as well.

(3) We first split up the integral into simple improper integrals:
\[
\int_{-\infty}^{\infty} e^{-t^2} \, dt = \int_{-\infty}^{0} e^{-t^2} \, dt + \int_0^{\infty} e^{-t^2} \, dt.
\] (3.21)

By part (2), the second integral on the right-hand side is equal to \(\frac{\sqrt{\pi}}{2}\). Because the function \(e^{-t^2}\) is even, the two integrals on the right-hand side are also equal, and thus the first integral on the right-hand side is also equal to \(\frac{\sqrt{\pi}}{2}\). Thus
\[
\int_{-\infty}^{\infty} e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} = \sqrt{\pi}.
\] (3.22)