1. Read:
   (a) Sections 10.1–10.3 (in either edition of the textbook)

2. Complete:
   (a) Exercises 1-15 below
   (b) Problems M and N below

    Unless otherwise specified, show your work to receive full credit.

**Exercises**

In exercises 1-6 below, compute the first four terms (i.e., up to \((x-a)^3\)) of the Taylor series for the given function \(f(x)\) about the given value of \(a\):

1. \(f(x) = (x+1)^{1/4}, \quad a = 0\)
2. \(f(x) = \ln(1-x), \quad a = 0\)
3. \(f(x) = \frac{1}{\sqrt{1+x}}, \quad a = 0\)
4. \(f(x) = \sin x, \quad a = \pi/4\)
5. \(f(x) = \tan x, \quad a = \pi/4\)
6. \(f(x) = \frac{1}{x}, \quad a = 2\)

By recognizing each series in exercises 7-9 below as a Taylor series evaluated at a particular value of \(x\), find the sum of each of the following convergent series:

7. \(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \ldots + \frac{(-1)^n}{(2n+1)!} + \ldots\)
8. \(1 - \frac{100}{2!} + \frac{10000}{4!} - \ldots + \frac{(-1)^n10^{2n}}{(2n)!} + \ldots\)
9. \(1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \ldots\)

10. Let \(i = \sqrt{-1}\). We define \(e^{i\theta}\) by substituting \(i\theta\) in the power series for \(e^x\). Use this definition to explain Euler’s formula

    \[ e^{i\theta} = \cos \theta + i \sin \theta. \]
In exercises 11-14 below, use multiplication and/or substitution with known Taylor series to find the Taylor series about 0 for the given function:

11. \( \ln(1 - 2y) \)
12. \( t \sin(3t) \)
13. \( \frac{z}{e^{z^2}} \)
14. \( \arctan(r^2) \)

15. (a) Find the Taylor series for \( f(t) = te^t \) about \( t = 0 \).
    (b) Using your answer to part (a), find a Taylor series expansion about \( x = 0 \) for
        \[ F(x) = \int_0^x te^t \, dt. \]
    (c) Using your answer to part (b), show that
        \[ \frac{1}{2} + \frac{1}{3} + \frac{1}{4(2!)} + \frac{1}{5(3!)} + \frac{1}{6(4!)} + \ldots = 1. \]

**Problem M**

Let \( p \) be any real number and let \( n \) be an integer, \( n \geq 0 \). We define the \((p,n)\)th binomial coefficient \( \binom{p}{n} \) (read “\( p \) choose \( n \)”) as follows:

- For any value of \( p \), \( \binom{p}{0} = 1 \).
- If \( n \geq 1 \), \( \binom{p}{n} \) is a product of \( n \) fractions with both the numerator and denominator decreasing by 1 in each factor:
  \[ \frac{p \cdot p - 1 \cdot p - 2 \ldots p - n + 2}{n \cdot n - 1 \cdot n - 2 \ldots 2} \cdot \frac{p - n + 1}{1} \]
  (Notice that this is similar to how factorials are defined: \( k! = k(k-1)(k-2)\ldots3 \cdot 2 \cdot 1 \).)

The reason it’s read “\( p \) choose \( n \)” is the following: If \( p \) is a positive integer, then \( \binom{p}{n} \) counts the number of ways one can select a(n unordered) group of \( n \) (distinct) objects from a set of \( p \) (distinct) objects. For example, the number of ways to choose 2 different letters from \( \{A, B, C, D\} \) is 6, since \( \binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6 \):
The six ways are \( AB, AC, AD, BC, BD, CD \).

a. Like factorials, binomial coefficients see a lot of action in probability, since they have a combinatorial (counting) interpretation. For example, you can use binomial coefficients to find the probabilities of various hands in poker (the card game). *To do so, the first thing you must do is compute the total number of possible hands—A poker hand is a selection of 5 cards from a standard 52 card deck.*

How many different possible poker hands are there? Express it as a binomial coefficient, and then compute that binomial coefficient (you will want a calculator for the second part).

b. Let \( p \) be a positive integer. Using both the mathematical definition of \( \binom{p}{n} \) above and the counting interpretation, explain why \( \binom{p}{n} = 0 \) if \( n > p \).
Another place where binomial coefficients appear is in the expansion of \((x + 1)^p\): If \(p\) is a natural number (\(p = 0, 1, 2, 3, \ldots\)), then
\[
(x + 1)^p = \sum_{n=0}^{p} \binom{p}{n} x^n
\]
(somewhat more detailed versions of the the formula above are referred to as the binomial theorem).

For example,
\[
(x + 1)^4 = \binom{4}{0} x^0 + \binom{4}{1} x^1 + \binom{4}{2} x^2 + \binom{4}{3} x^3 + \binom{4}{4} x^4 = 1 + 4x + 6x^2 + 4x^3 + x^4
\]
which is exactly what you would get if you took the time to expand \((x + 1)^4\) by the usual Algebra I methods.

Amazingly, this formula works even if \(p\) is not a natural number: For any \(p\), we have the power series representation
\[
(x + 1)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n \text{ on the interval of convergence.}
\]

c. Why are we allowed to make the upper bound infinity without affecting the original formula (*)? Let’s see, by example: Explain why \(\sum_{n=0}^{\infty} \binom{4}{n} x^n = \sum_{n=0}^{4} \binom{4}{n} x^n\).

d. Show that if \(\binom{p}{n} \neq 0\), then \(\binom{p}{n+1} / \binom{p}{n} = \frac{p-n}{n+1}\).

e. Using (d) find the radius of convergence of \(\sum_{n=0}^{\infty} \binom{p}{n} x^n\) when \(p\) when \(p\) is not a natural number (you do not need to test the endpoints). What is the RoC when \(p\) is a natural number?

f. Compute \((-1)^n\) for \(n = 0, 1, 2, 3, 4\). What will the expansion of \((x + 1)^{-1}\) look like? How else could we have obtained this particular power series representation?

g. Using the fact that
\[
\int \frac{dx}{\sqrt{1 - x^2}} = \int (1 - x^2)^{-1/2} dx = \arcsin x + C
\]
find a power series that represents the arcsine function on an interval centered at 0. Your answer should be in the form of an infinite power series, whose general term involves binomial coefficients (with \(n\) in the lower position). (The IoC, which you don’t need to find, will be \([-1, 1]\).)
Problem N

If $\sum a_n$ is alternating with $\{|a_n|\}$ (strictly) decreasing and tending to zero, then we have

$$\text{error in } M\text{th partial sum} = |S_\infty - S_M| = \left| \sum_{n=0}^\infty a_n - \sum_{n=0}^M a_n \right| \leq |a_{M+1}|$$

Using power series, each of the quantities below can be represented as an alternating series $\sum a_n$ that converges by the alternating series test. For each one, (i.) Write down that series; (ii.) Find the smallest $M$ such that $|a_{M+1}| < 10^{-4}$; (iii.) Write the decimal expansion of $S_m$ rounded to 8 decimal places; and (iv.) Use a calculator to find the “exact” value rounded to 8 decimal places.

a. $\sin(1)$

b. $\cos(1)$

c. $e^{-1/4}$

d. $\arctan(1/3)$

e. $(5/4)^{-1/2} = \frac{2}{\sqrt{5}}$. 