Information and instructions

This assignment covers power series representations and manipulations (the contents of Lectures 19, 22) and the Taylor series construction (Lecture 23).

All of it can be done without Taylor series, with the exception of the last problem. However, some other problems also require the power series for sine and cosine. This information will be covered in Lecture 23 (3/04) and is also covered in section 10.2 of the textbook if you want to read ahead.

Conceptual questions

1. We have the power series representation

$$\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{n+1}}{n+1} \quad \text{when} \quad 0 < x \leq 2$$

a. Show that the alternating harmonic series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \quad (*)$$

converges to $\ln 2$.

This is simply the result of plugging $x = 2$ into the series (note that this is valid, since the series converges at $x = 2$).

b. Show that the series

$$\sum_{n=0}^{\infty} \frac{1}{2n+1(n+1)} = \frac{1}{2} \cdot 1 + \frac{1}{2^2 \cdot 2} + \frac{1}{2^3 \cdot 3} + \frac{1}{2^4 \cdot 4} + \frac{1}{2^5 \cdot 5} + \cdots \quad (***)$$

also converges to $\ln 2$. Hint: This still uses the formula for $\ln x$, but it is not immediately obvious what value of $x$ to plug in. You'll need the log rule $\ln(1/A) = -\ln(A)$.

This takes a bit of guess-and-check but the two clues are (i) that there is $2^{n+1}$ in the denominator (suggesting $x - 1 = 1/2$ or $x - 1 = -1/2$) and (ii) that the alternation is no longer present (suggesting it was canceled out by $x - 1$ being negative). Indeed, if we plug in $x - 1 = -1/2$ (i.e., $x = 1/2$), we have

$$\ln(1/2) = \sum_{n=0}^{\infty} \frac{(-1)^n(-1/2)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{n+1}}{2^{n+1}(n+1)} = \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}(n+1)}$$

Multiplying the ends of this series by $-1$ and using log rules on the left, we have $\ln 2 = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)}$ as desired.

c. Using a calculator, compute the errors in the 9th partial sums of the two series $(\ast)$ and $(\ast\ast)$ above. That is, compute

$$\left| \ln 2 - \sum_{n=0}^{9} \frac{(-1)^n}{n+1} \right| \quad \text{and} \quad \left| \ln 2 - \sum_{n=0}^{9} \frac{1}{2^{n+1}(n+1)} \right|$$

How do these errors compare?

The errors are $\approx 0.0475$ and $\approx 0.0000823$. The error in $(\ast\ast)$’s 9th partial sum is way smaller (meaning the partial sum gives a much better estimate of $\ln 2$).

The “speed” at which a convergent series converges to its value depends on several factors, but two major ones are the type of convergence (absolute vs. conditional) and the value of $L$ one computes when performing the ratio test.

d. Determine whether series $(\ast)$ and $(\ast\ast)$ converge absolutely or conditionally, and then use this information to make a guess as to whether absolutely or conditionally convergent series converge more quickly.

$(\ast)$ converges conditionally (since the non-alternating harmonic series diverges) while $(\ast\ast)$ converges absolutely (since it converges and has only positive terms, or by ratio test). Generally speaking, absolutely convergent series converge more quickly than conditionally convergent series.

e. Determine the value of $L$ one would compute when performing the ratio test on series $(\ast)$ and $(\ast\ast)$, and then use this information to make a guess as to how the value of $L$ relates to the speed at which a series converges.

$(\ast)$ has $L = 1$ while $(\ast\ast)$ has $L = 1/2$. Generally, a series with a smaller $L$-value will converge faster. Convergent series where the Ratio Test is inconclusive (because $L = 1$) converge the slowest. Series with $L = 0$ such as $\sum \frac{1}{n!}$ converge very quickly.

**Routine problems**

2. Using the known power series for $\frac{1}{1-x}$, $e^x$, $\ln x$, and $\arctan x$, $\sin x$, and $\cos x$, write down power series that represent the following functions on their intervals of convergence (which you do not need to specify):

   a. $\frac{3x}{4x^2 - 1}$   b. $e^{-x^2/2}$   c. $x \cos x$   d. $\frac{\sin(x^3)}{x^2}$   e. $\frac{1 - \cos x}{x}$
\[ a. = (−3x) \cdot \frac{1}{1−4x^5} = (−3x)\sum_{n=0}^{∞}(4x^5)^n = −3\sum_{n=0}^{∞}4^n.x^{5n+1}. \]

\[ b. = \sum_{n=0}^{∞} \frac{(−x^2/2)^n}{n!} = \sum_{n=0}^{∞} \frac{(−1)^nx^{2n}}{2^n \cdot n!}. \]

\[ c. = x\sum_{n=0}^{∞} \frac{(−1)^nx^{2n}}{(2n)!} = \sum_{n=0}^{∞} \frac{(−1)^nx^{2n+1}}{(2n)!}. \]

\[ d. = \frac{1}{x^2}\sum_{n=0}^{∞} \frac{(−1)^n(x^3)^{2n+1}}{(2n+1)!} = \frac{1}{x^2}\sum_{n=0}^{∞} \frac{(−1)^nx^{6n+3}}{(2n+1)!} = \sum_{n=0}^{∞} \frac{(−1)^nx^{6n+1}}{(2n+1)!} \]

\[ e. = \frac{1}{x} \left(1−\frac{x^2}{2!} + \frac{x^4}{4!} − \frac{x^6}{6!} + \cdots\right) = \frac{x^2−x^4+x^6−}{x} = \frac{x}{2} − \frac{x^3}{4!} + \frac{x^5}{6!} − \cdots \]

\[ = \sum_{n=0}^{∞} \frac{(−1)^nx^{2n+1}}{(2n+2)!}. \]

3. Using the known power series for $\frac{1}{1-x}$, $e^x$, $\ln x$, and $\arctan x$, $\sin x$, and $\cos x$, write down convergent infinite series that converge to the following quantities:

\[ a. \quad b. \quad \ln(1/3) \quad c. \quad \arctan(1/239) \quad d. \quad \cos(5) \]

\[ e. \quad \int_0^{1} e^{-x^2/2} \, dx \quad f. \quad \int_{−1}^{1} \frac{\arctan x}{x} \, dx \]

Simplify your answers as much as possible.

\[ a. = \sum_{n=0}^{∞} \frac{1}{n!}. \]

\[ b. = \sum_{n=0}^{∞} \frac{(−1)^n(−2/3)^{n+1}}{n+1} = −\sum_{n=0}^{∞} \frac{2^{n+1}}{3^{n+1}(n+1)}. \]

\[ c. = \sum_{n=0}^{∞} \frac{(−1)^n}{239^{2n+1}(2n+1)}. \]

\[ d. = \sum_{n=0}^{∞} \frac{(−1)^n25^n}{(2n)!}. \]

\[ e. = \int_0^{1} \sum_{n=0}^{∞} \frac{(−1)^nx^{2n}}{2^n \cdot n!} \, dx = \sum_{n=0}^{∞} \frac{(−1)^n}{2^n \cdot n!} \int_0^{1} x^{2n} \, dx = \sum_{n=0}^{∞} \frac{(−1)^n}{2^n \cdot n! \cdot (2n+1)}. \]

\[ f. = \int_{−1}^{1} \sum_{n=0}^{∞} \frac{(−1)^nx^{2n}}{2n+1} \, dx = \sum_{n=0}^{∞} \frac{(−1)^n}{2n+1} \int_{−1}^{1} x^{2n} \, dx = 2\sum_{n=0}^{∞} \frac{(−1)^n}{(2n+1)^2}. \]
4. Using the bounds from the alternating series test, write down an estimate the value of the integral in part (e) to error $< 10^{-6}$. Your answer should be a simple alternating (finite) sum of fractions.

We have

$$
\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n! \cdot (2n + 1)} = 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} - \frac{1}{42240} + \frac{1}{599040} - \frac{1}{9676800} + \cdots
$$

The first term $< 10^{-6}$ is the last one written on the right hand side above, so our estimate is everything before that, namely:

$$1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} - \frac{1}{42240} + \frac{1}{599040}.$$

This gives an estimate of $\int_0^1 e^{-x^2/2} \, dx \approx 0.855624898$ while the “actual” value of the integral is $0.8556243919$. Our estimate is correct to six decimal digits, which is pretty good.

5. To what values do the following infinite series converge? Justify your answers.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}(n + 1)}$

b. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

c. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

d. $\frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{9^n(2n + 1)}$

a. This looks like the $\ln x$ series. Solving $(x - 1)^{n+1} = \frac{1}{5^{n+1}}$ gives $x - 1 = \frac{1}{5}$, so this series was obtained by plugging $x = 6/5$ into the log series. It converges to $\ln(6/5)$.

b. This is $e^{-1} = \frac{1}{e}$.

c. This is $\cos(1)$.

d. This is $\text{arctan}(1/3)$ — the simplification uses $\frac{1}{3^{2n+1}} = \frac{1}{3} \cdot \frac{1}{3^n}$.

6. The hyperbolic cosine is the function $\cosh x = \frac{e^x + e^{-x}}{2}$ and the hyperbolic sine is $\sinh x = \frac{e^x - e^{-x}}{2}$. Their power series are

$$
cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \text{and} \quad \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n + 1)!}
$$

(a)

a. The power series for hyperbolic sine and cosine superficially resemble those of the trigonometric (really the “circular”) sine and cosine. How?

The hyperbolic versions lack the alternation factor $(-1)^n$. 4
b. Prove the formulas in (†), using either manipulation of known power series or by constructing the Taylor series for these functions (centered at 0). For the latter, it may be useful to know that \((\sinh x)’ = \cosh x\) and \((\cosh x)’ = \sinh x\).

Using the table method is pretty straightforward: The pattern in the \(f^{(k)}(x)\) row will alternate between \(\cosh x\) and \(\sinh x\), so the next row will alternate between \(\cosh 0 = \frac{e^0 + e^0}{2} = 1\) and \(\sinh 0 = \frac{e^0 - e^0}{2} = 0\). So, the tables for these functions will look like those for sine and cosine, but with no alternation, hence the formulas in (†).

If you’re curious how to do these with series manipulation instead, here’s how it goes for hyperbolic cosine: Expand \(e^x\) and \(e^{-x}\) to get

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots
\]

\[
e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots
\]

Now, adding both lines above, the odd-powered terms cancel, so

\[
e^x + e^{-x} = 2 + 2 \cdot \frac{x^2}{2!} + 2 \cdot \frac{x^4}{4!} + \cdots
\]

and dividing by 2 gives the Taylor series for \(\cosh x\) in (†).

7. Suppose we have a mystery function \(f\) that satisfies \(f^{(n)}(0) = \frac{4^n}{n!}\). Write down a power series (centered at 0) that represents the function \(f\).

The coefficient of \(x^n\) in the Taylor series is \(\frac{f^{(n)}(0)}{n!}\), so the Taylor series for this mystery function at 0 is just

\[
\sum_{n=0}^{\infty} \frac{4^n x^n}{(n!)^2}
\]