Finally, some applications! We have gathered nine problems from a variety of fields that feature power series in some way. (Summarized in the table below.) I hope that this variety of applications paints a compelling picture for why we learned what we learned this quarter.

**Directions:** Read problems 1–9. Complete one of problems 1–4 and one of problems 6–9 for a total of two problems. Indicate clearly on your assignment which problems you did!

For each problem you only have to write something down in response to those subproblems with a (*), (**), or (†); (**) indicates that you will have to use a calculator to complete the subproblem, and (†) indicates that you will have to look something up (the answers should be fairly evident after one or two Google/Wikipedia searches).

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1. **The Basel problem and the zeta function**

The Basel problem, posed in 1644 by Italian mathematician Pietro Mengoli, was the problem of finding the exact value of the convergent infinite series

$$
\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots
$$

$$
= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots
$$

The solution was given 90 years later by Swiss mathematician Leonhard Euler (after whom the constant $e$, among many other things, is named), in 1734. He wasn’t able to provide a rigorous proof until until 1741. Imagine: a simply posed problem that took almost a century to solve!

The proof below is not Euler’s original, which was based on another method using infinite series (as well as infinite products, which are like infinite series but with multiplication instead of addition). The proof below is also not the shortest proof, which uses multivariable integration (the topic of Math 52).
Let's get on with it! Because \( \sum \frac{1}{n^2} \) converges absolutely, we can reorder the terms:

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots
\]

\[
= \left( \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots \right) + \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right)
\]

\[
= \frac{1}{2^2} \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) + \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right)
\]

\[
= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}
\]

a.* Explain briefly what happens in each of the three steps above (how to get from the first line to the second, etc).

b.* Let \( \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} \). Rearrange the conclusion of (a.) to show that \( \zeta(2) = \frac{4}{3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \).

c. As you found on HW6, the Taylor series for inverse sine is given by

\[
\arcsin x = \sum_{n=0}^{\infty} \left( \frac{-1}{n} \right)^n \frac{(-1)^n x^{2n+1}}{2n+1}
\]

which converges on \([-1, 1]\)

The next step in our proof is to plug in \( x = \sin \theta \) above and integrate the whole thing!

\[
\int_{0}^{\pi/2} \theta d\theta = \int_{0}^{\pi/2} \arcsin(\sin \theta) d\theta = \int_{0}^{\pi/2} \sum_{n=0}^{\infty} \left( \frac{-1}{n} \right)^n \frac{(-1)^n (\sin \theta)^{2n+1}}{2n+1}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{-1}{n} \right)^n \frac{(-1)^n}{2n+1} \int_{0}^{\pi/2} (\sin \theta)^{2n+1} d\theta
\]

The definite integrals inside the series can be evaluated in terms of binomial coefficients like so:

\[
\int_{0}^{\pi/2} (\sin \theta)^{2n+1} d\theta = \frac{(-1)^n}{(-1)^n (2n+1)}
\]

d.* Use (c.) to show that \( \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} \).

e.* Finally, use (a–d) to compute \( \zeta(2) \).

f.† How did the Basel problem get its name?

The zeta function \( \zeta(s) = \sum \frac{1}{n^s} \) has been studied in depth since the mid 18th century. It is the subject of one of the most famous unsolved problems of mathematics, posed in 1859.

g.† What is the name of the unsolved, million-dollar problem concerning the properties of \( \zeta(s) \)?

1Fun, possibly disturbing fact: The terms of a conditionally convergent series cannot be arbitrarily reordered without changing the series’ value.
2. The Poisson distribution

In statistics and probability, *the number of random, independent occurrences of the same event within a fixed interval* are frequently modeled using a *Poisson process*. Examples:

- Suppose that, on average, 6 births occur at Stanford Hospital per day. Obviously, some days will have fewer births, and some days will have more. Given \( n \), what is the probability that during a given day, Stanford Hospital assists in the birth of \( n \) babies?

- Approximately 0.56 meteorites weighing 10 grams or more strike the Earth every hour (on average). What is the probability no such meteorites fall to the Earth on a given hour?

- Suppose that trees in a forest are more or less evenly distributed, with an average of 720 trees per acre. What is the probability that a randomly chosen acre of the forest has \( \geq 800 \) trees?

If one models this sort of situation carefully, one arrives at the *Poisson distribution*: A Poisson random variable \( X \) “with parameter \( \lambda \) (\( \lambda > 0 \)), which counts events in a Poisson process, is a variable that can take nonnegative integer values \( 0, 1, 2, \ldots \) . It takes the value \( n \) with probability

\[
\Pr(X = n) = \frac{\lambda^n e^{-\lambda}}{n!}
\]

a.* One of the axioms (unbreakable rules) of probability is that the sum of all probabilities should add up to 1. For the Poisson distribution this means we must have

\[
\sum_{n=0}^{\infty} \Pr(X = n) = \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} = 1
\]

Prove the second equality above, using power/Taylor series.

b.* The average (or expected) value of a Poisson random variable is given by

\[
\sum_{n=0}^{\infty} n \cdot \Pr(X = n) = \sum_{n=0}^{\infty} \frac{n\lambda^n e^{-\lambda}}{n!}
\]

Evaluate the above series.

*Hint: It starts like this.*

\[
\sum_{n=0}^{\infty} \frac{n\lambda^n e^{-\lambda}}{n!} = \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!}
\]

c.* In light of (b.), what values would you choose for \( \lambda \) in the hospital, meteorite, and forestry situations described before problem (a.)?

d.** What are the chances that exactly 3 babies are born at Stanford Hospital on a given day? Fewer than 4 babies? More than 3 babies? What are the chances zero meteorites (of 10 grams or more) fall in a given hour?

e.† After whom is the Poisson distribution named? What were they studying when they formulated the Poisson distribution?
3. The Gibbs phenomenon

If you’ve ever converted a high resolution image file into the JPG format, you might have noticed that the compression process produces artifacts or distortion near boundaries in the picture. Why does this still happen? It’s 2017, for Pete’s sake!

![Fig. 3.1. This star looks terrible, just awful.](image)

The reason some of these artifacts—“ringing artifacts” in particular—can happen is due to something called the Gibbs phenomenon. Skipping the technical details, the *Fourier series*

\[
\frac{4}{\pi} \left( \sin x + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots \right)
\]

converges to the *square wave* of amplitude 1 and period \(2\pi\) radians, but near the jump discontinuities of the square wave, the partial sums of the Fourier series above “overshoot” the actual square wave:

![Fig. 3.2. Fourier approximations to the square wave exhibit an overshoot near the jump discontinuities (the overshoot is \(\approx\) the extra height of the last peak in the red circle).](image)

This overshoot is responsible for the image artifacts we love to hate! In addition to the aesthetic problems, the Gibbs phenomenon can lead to misdiagnoses because of its role in medical imaging:

\[\text{Fourier series are the subject of section 10.5. We are not covering them this quarter, so you don’t need to know anything about them for the final. In short, they are like power series but with } \sin(nx) \text{ and } \cos(nx) \text{ in place of } x^n.\]
Fig. 3.3. The Gibbs phenomenon is responsible for the “ripples” in this MRI.

a.* Use Taylor series to evaluate $\lim_{x \to 0} \left( \frac{\sin x}{x} \right)$.

b.* Write down an infinite series (not a power series! no variables) whose value is equal to

$$\int_0^\pi \frac{\sin x}{x} \, dx$$

Technical note: Because $\frac{\sin x}{x}$ does not have an asymptote at $x = 0$, the integral above is not “really” improper, even if the integrand is not literally defined at the left endpoint. In practice, this particular kind of discontinuity—a “removable” discontinuity—can be ignored for most purposes by “filling in” the missing value of the function, here defining $\frac{\sin 0}{0}$ to be the value of the limit from (a.).

c.** Estimate the value of the integral in (a.) by evaluating $\int_0^\pi P_8(x) \, dx$ where $P_8(x)$ is the 8th degree Taylor polynomial for $\frac{\sin x}{x}$. You should use the approximation $\pi \approx 3.14159$ in your answer, or something more precise (3.14 won’t be good enough).

d.** What does this integral have to do with the Gibbs phenomenon? Well, the amount by which the Fourier approximation overshoots the square wave (height of the last peak near the jump discontinuity) will tend to

$$\frac{1}{\pi} \int_0^\pi \frac{\sin t}{t} \, dt - \frac{1}{2}$$

percent of the square wave’s amplitude. Estimate the above by replacing the integral above with your answer in (c.) and using $\pi \approx 3.14159$.

Note: Knowing how badly the Fourier approximations overshoot the square wave is the first step in filtering such artifacts out.

e.† It’s usually called the Gibbs phenomenon, but who actually discovered it, and when?

f.† What is the name of the spinal disorder whose presence in an MRI can resemble artifacts from the Gibbs phenomenon?

g.† Experimental physicists were seeing the Gibbs phenomenon in instrumental readings by the mid-19th century, but they assumed that its effects were actually due to what?
4. The circumference of an ellipse

You are all doubtlessly familiar with the formula for the circumference of a circle: \( C = 2\pi r \) where \( r \) is the radius of the circle. Surprisingly, there is no “nice” formula for the circumference of an ellipse in terms of its major and minor radii.

Fig. 4.1. The most enigmatic member of the shapes kingdom—the ellipse.

A suitably advanced calculator (i.e., Google) might be able to estimate elliptic circumferences for you, but how can it do this?

If you have any experience with parametric equations for circles and ellipses (such as in Math 20 last quarter), you may remember that, as \( \theta \) varies from 0 to \( 2\pi \), the point \((a \cos \theta, b \sin \theta)\) traces an ellipse in the \( xy\)-plane with center \((0, 0)\), \( x\)-radius \( a \), and \( y\)-radius \( b \). For ease of exposition, we will assume that \( a \leq b \) so that \( a \) is the minor radius and \( b \) is the major radius.

The arclength formula for parametric curves (this was also in Math 20) says that our ellipse has circumference equal to

\[
C = \int_{0}^{2\pi} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta
\]

Using \( \cos^2 \theta = 1 - \sin^2 \theta \) and doing some algebra yields

\[
C = b \int_{0}^{2\pi} \sqrt{1 - x \sin^2 \theta} \, d\theta
\]

where \( x = 1 - \frac{a^2}{b^2} \). We can now replace the integrand with the binomial series for \((1 + y)^{1/2}\), substituting \( y = -x \sin^2 \theta \):

\[
C = b \int_{0}^{2\pi} \sum_{n=0}^{\infty} \left(\frac{1/2}{n}\right) (-1)^n x^n (\sin \theta)^{2n} \, d\theta = b \sum_{n=0}^{\infty} \left(\frac{1/2}{n}\right) (-1)^n x^n \int_{0}^{2\pi} (\sin \theta)^{2n} \, d\theta
\]

And finally, using the formula \( \int_{0}^{2\pi} (\sin \theta)^{2n} \, d\theta = 2\pi (-1)^n \left(\frac{-1/2}{n}\right) \) and simplifying,

\[
C = 2\pi b \sum_{n=0}^{\infty} \left(\frac{1/2}{n}\right) \left(\frac{-1/2}{n}\right) x^n
\]

where, again, \( x = 1 - \frac{a^2}{b^2} \). Let \( F(x) \) be the power series above so that an ellipse with minor radius \( a \) and major radius \( b \) has circumference equal to \( 2\pi b F \left(1 - \frac{a^2}{b^2}\right) \).

a. Note that \( F(0) = \left(\frac{1/2}{0}\right) \left(\frac{-1/2}{0}\right) = 1 \), so when \( a = b \) the circumference is equal to \( 2\pi b F(0) = 2\pi b \). This is because when \( a = b \) the ellipse is a plain ol’ circle with radius \( b \) (!). Nice.

\[\text{This ellipse can also be graphed with the implicit equation } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.\]
b. * Using the fact that 
\[
\frac{\binom{p}{n+1}}{\frac{p_n}{p}} = \frac{p - n}{n + 1}
\]
find \(F(x)\)'s radius of convergence. (It will converge at the endpoints; you do not need to check this!)

c. * One way we can estimate elliptic circumferences is to expand a couple of terms of \(F(x)\). Expand \(F(x)\) up to the \(x^4\) term. That is, find the 4th Taylor polynomial for \(F(x)\). Simplify the binomial coefficients so that the coefficients of the polynomial you write down ain't nothing but good ol' fractions.

d. ** Use the above to estimate the circumference of an ellipse with major radius 2 and minor radius 1. How does this compare with \(4\pi\) (twice the circumference of a circle with radius 1)?

e. ** The orbits of most objects in the solar system are ellipses (some comets have hyperbolic orbits, which means they only pass through once).

The minor and major radii of Pluto’s orbit are 3.55 billion miles and 3.67 billion miles, respectively. If Pluto’s year is 248 earth years, then what is Pluto’s average speed (with respect to a fixed Sun) in miles per hour? (Use your Taylor polynomial from c. to estimate this!)

f. * Fix \(b = 1\). As \(a \to 0^+\), \(1 - \frac{a^2}{b^2} \to 1^-\), and the ellipse approaches the degenerate (or flat) ellipse with major radius 1 and minor radius 0. Since this ellipse “has circumference 4,” we obtain the formula \(4 = 2\pi F(1)\).

![Diagram of ellipse deformation]

In light of the above argument, find the exact value of the series
\[
F(1) = \sum_{n=0}^{\infty} \binom{1/2}{n} \binom{-1/2}{n} = 1 - \left( \frac{1}{4} + \frac{3}{64} + \frac{5}{256} + \frac{175}{16384} + \cdots \right)
\]

Note: I didn’t know the series above converges to a nice value until I wrote up this problem set. Never stop exploring and learning new things!

g. † Historically, the problem of finding the circumference of an ellipse led to the development of modular forms and elliptic curves, which have found both abstract and practical applications.

What famous problem (posed in 1637) was solved in 1994 using elliptic curves? In what branch of computer security do elliptic curves find practical applications?

---

4 Full disclosure: I’m biased because my own research is on modular forms. I don’t just do calculus all day, fortunately!
5. The three-body problem

In a celestial system with two objects, gravitation works very simply: The objects attract each other by a force inversely proportional to the square of the distance between them. One can use this fact to reckon that planetary orbits are typically elliptical and that a planet orbits its star once every fixed period (a year). Because of these facts, one can model a two-body system pretty much exactly!

However, the situation becomes much, much, much more complicated when studying the gravitational interaction of three (or more) celestial bodies. There are special situations in which the bodies will orbit each other periodically, but most set-ups lead to extremely chaotic behavior. Fig. 5.1. Some 3-body simulations: one periodic and two chaotic.

The three body problem was first formalized in the 1740s (though Galileo and Newton both considered it in their own studies of the cosmos). Though no simple general solution is known even today, the Finnish mathematician Karl Sundman found a solution in terms of power series in 1912. Because Sundman’s series are in terms of \( x = t^{1/3} \) (where \( t \) is time), they converge rather slowly, and because of this, they are of little practical use. In fact, it is estimated that in order to use them for astronomical observations, one would need to compute the series to \( 10^{8000000} \) terms. Yikes!

You are probably wondering how this could be true, given that we can model the solar system pretty accurately: There are far more than two celestial bodies in the solar system! The short, slightly misleading answer is that the sun is so large that the effect of its gravity “drowns out” significant gravitational interactions between the planets, which are much less massive.

6. The Bessel function

If you did practice problems to study for Midterm 2 you may recall a textbook problem on the Bessel function (9.5.45), which is defined in terms of a power series:

\[
J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2}
\]

converging on \((-\infty, \infty)\).

How was the Bessel function discovered? It turns out \( y = J_0(x) \) is a solution to the differential equation \( x^2 y'' + xy' + x^2 y = 0 \) with initial conditions \( y(0) = 1 \) and \( y'(0) = 0 \). Here’s a brief explanation of how this is derived (with some input from you!).

\[ \text{a.} \quad \text{Expand } J_0(x) \text{ to the } x^4 \text{ term.} \]

\[ \text{5The science fiction novel The Three-Body Problem by Cixin Liu, which won the Hugo Award in 2015, employs the chaotic interactions between three celestial bodies as a central plot element. It’s a fun read, too.} \]
b. Suppose that \( y = F(x) \) is a solution to \( x^2y'' + xy' + x^2y = 0 \), where \( F(x) \) is a power series
\[
F(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots
\]
If \( F(0) = 1 \) and \( F'(0) = 0 \), then what are \( c_0 \) and \( c_1 \)?
c. Manipulating \( F(x) \), we have
\[
x^2F''(x) = 2c_2x^2 + 6c_3x^3 + 12c_4x^4 + \cdots
\]
\[
xF'(x) = c_1x + 2c_2x^2 + 3c_3x^3 + 4c_4x^4 + \cdots
\]
\[
x^2F(x) = c_0x^2 + c_1x^3 + c_2x^4 + \cdots
\]
So if \( F(x) \) solves \( x^2y'' + xy' + x^2y = 0 \), then we must have
\[
x^2F''(x) +xF'(x) +x^2F(x) = c_1x + (4c_2 + c_0)x^2 + (9c_3 + c_1)x^3 + (16c_4 + c_2)x^4 + \cdots = 0
\]
Use this information (plus part (b.)) to solve for \( c_2 \), \( c_3 \), and \( c_4 \).
d. Compare your answers in (a.) and (c.) and smile in a secret, satisfied way to yourself.
e. Estimate \( J_0(2) \) to 4 decimal places.

With a bit more work (like a–c but with more variables) one can derive the Bessel function in its entirety from the differential equation it was born to solve. This is a major application of power series: Solving differential equations and initial value problems that cannot be solved by the “elementary” method of separating variables and integrating.

f. The Bessel function plays a role in modeling the vibrations of which musical instrument?
g. There is famous equation in quantum mechanics named after a German physicist where Bessel functions assist in finding solutions. What is the name of this equation? Hint: A hypothetical cat is also named after this physicist.

7. The speed of ocean waves

My high school physics teacher once gave us a problem set on ocean waves (they were apparently the topic of his doctoral dissertation). Something that struck me about this problem set was that there were two equations for the speed of an ocean wave:

- In shallow water, \( v = \sqrt{gd} \) where \( g \) is gravity and \( d \) is average depth.
- In deep water, \( v = \sqrt{gL/2\pi} \) where \( g \) is gravity and \( L \) is the wavelength (distance between peaks of waves).

It turns out there is a perfectly reasonable explanation for this dichotomy: The actual equation for ocean wave speed is
\[
v = \sqrt{\frac{gL}{2\pi} \tanh \left( \frac{2\pi d}{L} \right)}
\]
where \( \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \) is the hyperbolic tangent function.
a.* Find the degree 1 Taylor polynomial of \( \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \).

b.* Find the limit \( C \) of \( \tanh x \) as \( x \to \infty \).

c.* Graph \( y = P_1(x) \), \( y = C \), and \( y = \tanh x \) on the same plot.

d.* Explain why when \( d \) is small (i.e., in shallow water), \( v \) is approximately \( \sqrt{gL} \).  

\[ \sqrt{gL} \cdot \frac{2\pi d}{L} = \sqrt{gd}. \]

e.* Explain why when \( d \) is large (i.e., in deep water), \( v \) is approximately \( \frac{gL}{2\pi} \).

The difference between the shallow and deep water equations also explains why waves break near the shore (but don’t do so out in open water): Near shore, the top of the wave is moving faster (since the water is deeper relative to the top) than the bottom, so the top eventually spills over and the wave breaks! So—think on the hyperbolic tangent the next time you go surfing.

8. Quartz clocks

Without clocks, we would be no better than wild, tardy beasts. But how do clocks even work?

![Image of a quartz clock](image)

**Fig. 8.1.** There’s a tiny piece of vibrating quartz in there. It says so on the face.

If you have a (non-smartphone) clock or watch and you look closely at its face, you may spot QUARTZ written in small text somewhere. Why? Well, inside this particular kind of clock is a tiny quartz crystal that oscillates back and forth quickly (when subjected to an electrical charge). The clock times ticks of its hands by counting the vibrations of the crystal! Pretty cool.

To calculate how many crystal oscillations there are per second (the fundamental frequency of a cantilever), one uses the equation

\[ f = \frac{\kappa^2}{2\pi} \cdot \frac{a}{l^2} \cdot \sqrt{\frac{E}{12\rho}} \]

where \( a \) is the crystal’s thickness, \( l \) is its length, \( E \) is its Young’s modulus (its elasticity), and \( \rho \) is its density. This leaves out \( \kappa \), the subject of this problem. \( \kappa \) is the smallest positive solution to the equation \( \cos(x) \cosh(x) = -1 \) where \( \cosh(x) \), the hyperbolic cosine, is defined by \( \cosh(x) = \frac{e^x + e^{-x}}{2} \).

a.* Using the table method, find the Taylor series expansion of \( \cosh x \), centered at zero as a \( \sum \) with its general term. How does your answer compare with the Taylor series for \( \cos x \)?

---

\[ \text{Wikipedia says that for a quartz crystal of length 3 mm and thickness 0.3 mm has a fundamental frequency of 33 kilohertz, so it will oscillate roughly 33000 times per second.} \]
b*. Let \( P_4 \) be the 4th Taylor polynomial for \( \cos x \) and let \( Q_4 \) be the 4th Taylor polynomial for \( \cosh x \). Find the exact value of the smallest positive solution of \( P_4(x)Q_4(x) = -1 \) (Same as solving \( P_4(x)Q_4(x) + 1 = 0 \)).

Note: You will be solving an octic (degree 8) equation here, which you’ve probably never done. However, if you did (a.) correctly and you expanded \( P_4 \cdot Q_4 \) correctly, you should be solving something like \( ax^8 + bx^4 + c = 0 \). This allows you to use the quadratic formula to solve for \( x^4 \), and then you can just take 4th roots to derive the final answer.

c.** Estimate your answer from (b.) to three decimal places. (Not using a series, just use a calculator.)

d.† How does your estimate from (c.) compare to the actual value of \( \kappa \) used in the Wikipedia article on quartz clocks?

e.† The invention of quartz clocks had serious negative ramifications for the watchmaking industry of which country (famous for its watches)?

9. Estimating \( \pi \)

The number \( \pi \) plays a central role in every scientific field, and precision measurements often depend on having very good approximations to \( \pi \). Because \( \pi \) is an irrational number, it cannot be determined exactly—its decimal expansion goes on and on forever and there can be no “pattern” among the digits.

Our species has spent a good deal of time and effort on the problem of estimating \( \pi \), starting with basic estimates like \( 3 \) (Babylon), \( \frac{256}{81} \) (Egypt), and \( \frac{22}{7} \) (Greece). This last approximation was found by Archimedes, whose method for approximating \( \pi \) was the state of the art for millennia:

\[ P_n = 2 \sin \left( \frac{\pi}{6 \cdot 2^n} \right) \]

which is the length of a side of the regular polygon with \( 6 \cdot 2^n \) sides. Since \( \sin x \approx x \) when \( x \) is close to zero, as \( n \to \infty \),

\[ 3 \cdot 2^n \cdot P_n = 6 \cdot 2^n \sin \left( \frac{\pi}{6 \cdot 2^n} \right) \to \pi \]
The ingenuity of Al-Kashi’s method was his discovery of a recursive formula for \( \{P_n\} \):

\[
P_0 = 1 \quad \text{and} \quad P_{n+1} = \sqrt{2 - \sqrt{4 - P_n^2}}
\]

For example, \( 6P_1 = 12 \sin \left( \frac{\pi}{12} \right) = 6\sqrt{2} - \sqrt{3} \approx 3.106 \).

Essentially, Al-Kashi used this recurrence to compute \( P_N \) for some large value of \( N \), (taking square roots easier than computing sines back then) and then he approximated \( \pi \approx 3 \cdot 2^N \cdot P_N \).

a.** Using guess-and-check and a calculator, what is the first value of \( N \) that gives you an approximation \( \pi \approx 6 \cdot 2^N \sin \left( \frac{\pi}{6 \cdot 2^N} \right) \) accurate to 6 decimal places?

The next significant leap in \( \pi \) estimation technology came from power series.

b. The Taylor series for \( \arctan(x) \) centered at zero converges on the interval \([-1, 1]\). By plugging in \( x = 1 \), one obtains the elegant formula

\[
\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots
\]

but this formula is not very practical for estimating \( \pi \) because it converges very slowly. Why does it converge so slowly? **Hint:** On an earlier homework assignment, you showed that the alternating harmonic series converges to \( \ln(2) \) but very slowly. The above series converges slowly for the same reason.

c.† Who discovered the series in (b)? **Note:** This series was actually discovered 300 years before Taylor was born! The discoverer of this formula also found a much more quickly converging series representation for \( \pi \) in addition to the elegant one above, but their record was beaten soon after by al-Kashi.

d.** In the 17th century, John Machin used the tangent addition formula (an obscure trig identity) to obtain the formula \( \pi = 16 \arctan \left( \frac{1}{5} \right) - 4 \arctan \left( \frac{1}{239} \right) \). William Shanks used this formula and the Taylor series for \( \arctan(x) \) to compute \( \pi \) to 527 decimal places(!) in 1873.

What value(s) of \( x \) do you think Shanks plugged into \( \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \)?

e.** Estimating \( \pi \) using Shanks’ method requires far fewer terms than the “elegant” formula from (b). Explain why.

f.† What is the current record for computing digits of \( \pi \)?