Failure to follow the instructions below is a breach of the Stanford Honor Code:

• You may not use or consult any book or notes during the exam.*
• You may not use a calculator or the calculator function on any electronic device during the exam.*
• You may not access any internet-capable electronic device during the exam,* including smartphones and smartwatches, for any reason. These devices must be switched to “airplane mode” and disconnected from all wireless networks (both cellular and wifi) during the exam*.
• You must sit in your assigned seat.
• You may not communicate with anyone other than the course staff during the exam,* or look at anyone else’s solutions.

*“During the exam” is defined as: After you start the exam, and before you turn in the exam and leave the testing site.

• You have 90 minutes to complete this exam. If the course staff must ask you to stop writing or to turn in your exam more than once after time is called, you may receive a score of zero.
• After you have turned in your exam, you may not discuss the contents of this exam with ANYONE other than the course staff until 9:00 PM tonight.
• If you need to make a phone call during the exam, ask a proctor.

I understand and accept these instructions. All smart devices on my person are in airplane mode and disconnected from all wireless networks.

Signature: _______________________________________________________

Remember to show your work and justify your answer if required (additional tips are on the next page). Present all solutions in as organized a manner as possible. GOOD LUCK!
Here are some tips:

- If you have time, it’s always a good idea to check your work when possible.
- If you get the wrong answer but show your work, you have a better chance of receiving partial credit.
- DO NOT attempt to estimate any of your answers as decimals. For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is exact.
- The very last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

**Integral table entries you may need**

\[
\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C \\
\int \frac{du}{(u - r)(u - s)} = \frac{1}{r - s} \ln \left| \frac{u - r}{u - s} \right| + C
\]

\[
\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C
\]

\[
\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \left( \frac{u}{a} \right) + C
\]

In the entries above, $a, r, s$ are constants such that $a \neq 0$ and $r \neq s$.

**Values of arcsine and arctangent**

The table below gives important values of the arcsine and arctangent functions:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{1}{\sqrt{3}}$</th>
<th>$\frac{\sqrt{3}}{2}$</th>
<th>1</th>
<th>$\sqrt{3}$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>arcsin $x$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/3$</td>
<td>$\pi/4$</td>
<td>$\pi/2$</td>
<td>undef.</td>
<td>undef.</td>
</tr>
<tr>
<td>arctan $x$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/3$</td>
<td>$\pi/4$</td>
<td>$\pi/2$</td>
<td>$\pi/3$</td>
<td>$\pi/2$</td>
</tr>
</tbody>
</table>

For negative values: $\arcsin(-x) = -\arcsin(x)$ and $\arctan(-x) = -\arctan(x)$.

Blank entries in the table are not “nice” multiples of $\pi$. 

2
Unless otherwise specified, in problems 1–6 you do not need to justify your answer.

1. In each space of the table below, write a C if the integral in that column CONVERGES for the value of $p$ in that row. (If the integral diverges, you may leave it blank.)

<table>
<thead>
<tr>
<th></th>
<th>$\int_0^1 \frac{1}{x^p} , dx$</th>
<th>$\int_3^\infty \frac{1}{x^p} , dx$</th>
<th>$\int_1^\infty (1/x^p)^2 , dx$</th>
<th>$\int_0^1 \frac{1}{x^p + 1} , dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $p = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b. $p = 1/3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c. $p = 2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d. $p = 4/5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Complete the following statement: If $a > 0$, then $\int_0^\infty e^{-ax} \, dx$ is equal to...

3. Let $f(x)$ be a positive, continuous function.

   In parts (a) and (b), circle the correct answer. Part (c) is at the top of the next page.

   a. If $f(x)$ decays (i.e., if $\lim_{x \to \infty} f(x) = 0$), then $f(x)^2$ decays...

      
      - FASTER
      - SLOWER
      - NEED MORE INFO ABOUT $f(x)$

   b. If $f(x)$ grows without bound (i.e., if $\lim_{x \to \infty} f(x) = +\infty$), then $f(x)^2$ grows...

      
      - FASTER
      - SLOWER
      - NEED MORE INFO ABOUT $f(x)$
c. We say that \( f(x) \) is \textit{bounded away from zero} if there is a constant \( c > 0 \) such that \( f(x) \geq c \) for all \( x \geq 1 \). Justify the following statement:

If \( f(x) \) is bounded away from zero, then \( \int_{1}^{\infty} f(x) \, dx \) must diverge.

Your explanation doesn’t need to be overly long or technical, but it should be convincing.

4. Partario is attempting to determine whether the integral \( \int_{0}^{2} \frac{h(x)}{x} \, dx \) converges or diverges, where \( y = h(x) \) is graphed on the next page.

Partario decides to use direct comparison to show that the integral \textit{diverges}. His argument is below:

i. From the graph, we see that \( 0 \leq h(x) \leq 4 \) whenever \( 0 \leq x \leq 2 \).

ii. Therefore, \( 0 \leq \frac{h(x)}{x} \leq \frac{4}{x} \) whenever \( 0 < x \leq 2 \).

iii. Integrating yields \( 0 \leq \int_{0}^{2} \frac{h(x)}{x} \, dx \leq \int_{0}^{2} \frac{4}{x} \, dx \).

iv. The integral \( \int_{0}^{2} \frac{4}{x} \, dx = 4 \int_{0}^{2} \frac{1}{x} \, dx \) diverges...

v. ...so we may conclude that \( \int_{0}^{2} \frac{h(x)}{x} \, dx \) diverges too.

Partario’s argument is incorrect!

a. Which step(s) in his argument contains invalid reasoning?

\textit{Circle the correct answer.}

i. ii. iii. iv. v.
b. Determine whether the integral \( \int_{0}^{2} \frac{h(x)}{x} \, dx \) converges or diverges, and give a correct argument; OR, state that there is not enough information about \( h(x) \) to do so.

5. In (a–h) below, fill in the blank with the correct asymptotic relation \( \prec, \sim, \text{ or } \succ \).

You do not need to justify your answers.

a. \( x^{1/5} \quad \_\_\_\_\_ \quad x^{-5} \)   \quad b. \( x^{5} \quad \_\_\_\_\_ \quad (2x^{2} + 4)^{3} \)   \quad c. \( x + e^{x} \quad \_\_\_\_\_ \quad x^{2} + e^{x} \)   \quad d. \( e^{x} \quad \_\_\_\_\_ \quad (2^{x})^{2} \)

  e. \( e^{-x} \quad \_\_\_\_\_ \quad \frac{x}{3^{x}} \)   \quad f. \( x + e^{-x} \quad \_\_\_\_\_ \quad x^{2} + e^{-x} \)   \quad g. \( x^{0.001} \quad \_\_\_\_\_ \quad \ln(x) \)   \quad h. \( x \quad \_\_\_\_\_ \quad \frac{x^{5}}{(x^{2} + 1)^{2}} \)
6. The *hyperbolic secant* function is defined by \( \text{sech}(x) = \frac{2}{e^x + e^{-x}} \) for all real numbers \( x \).

This function and its inverse, \( \text{arcsech}(x) \), (which is defined when \( 0 < x \leq 1 \)) are graphed below:

Above, \( y = \text{sech}(x) \) is the “bell-shaped” curve, while \( y = \text{arcsech}(x) \) is the curve with a vertical asymptote. Note that the curves in the right quadrant are each other’s reflections over the line \( y = x \). This is because \( \text{arcsech}(x) \) is the inverse of \( \text{sech}(x) \).

a. Briefly explain why \( \int_0^\infty \text{sech}(x) \, dx = \int_0^\infty \frac{2}{e^x + e^{-x}} \, dx \) converges.

b. Suppose a friendly robot informs you that \( \int_{-\infty}^{\infty} \text{sech}(x) \, dx = 2\pi \). Using this information, evaluate \( \int_0^1 \text{arcsech}(x) \, dx \). *No justification required.*
In problems 7–9, evaluate the integral. Show your work. If you believe that the integral diverges, write DIVERGES as your final answer. Show all work and draw boxes around your final answers.

7. \[ \int_{1}^{\infty} x^2 e^{-x^3} \, dx \]

8. \[ \int_{1}^{\infty} \frac{4}{x^2 + 4x} \, dx \quad \text{Hint:} \quad \frac{4}{x^2 + 4x} = \frac{1}{x} - \frac{1}{x + 4}. \]
In problems 10–11, determine whether the integral **CONVERGES** or **DIVERGES** using the method of your choice. Be sure to justify your answer. Some tips:

- If you use limit or direct comparison, show enough work that we can follow your argument and understand how you came to your conclusion.
- You do not have to take limits step-by-step to verify asymptotic relations.
- You may use the convergence conditions for \( \int_0^1 \frac{1}{x^p} \, dx \), \( \int_1^\infty \frac{1}{x^p} \, dx \), \( \int_0^\infty e^{-ax} \, dx \), and \( \int_0^\infty r^{-z} \, dx \) without further justification, but mention when you use them.

**Draw boxes around your final answers.**

10. \( \int_0^\infty \frac{x^{100}}{100^x} \, dx \)
11. \( \int_{0}^{\infty} \frac{dx}{\sqrt{x^5 + 2x^3 + x}} \)
The remaining space is provided for any extra work. If you think this work is important to one of your solutions, please indicate that on the page of the relevant problem (otherwise we won’t know!).

If you need additional paper, ask a proctor.