Midterm 1 Expectations

Midterm 1 will be a 90-minute exam covering improper integrals, limit comparison (including growth/decay of functions), and geometric sums and series. This corresponds roughly to sections 7.6, 7.7, and 9.2 of the textbook (either addition), with some additional material that can be found in the lecture notes on asymptotic analysis and limit comparison—this is also addressed below.

The exam will consist both of short-answer and long-answer questions. The practice exams are fairly representative of the format.

Prerequisites

Math is a naturally cumulative subject so there are several important things that you are expected to know well from previous classes. You will not be tested directly on this material, but you are assumed to be familiar with it:

- You should be able to find the graph of a line given two points on that line, or the slope of the line and a single point.
- You should be able to factor and expand simple polynomials.
- You are expected to be familiar with basic rules for powers, radicals, exponents, and logarithms.
- You are expected to know how the various trig functions are defined (in terms of angles and right triangles) and their values at integer multiples of $\pi/2$. For Math 21 you will not have to know any trig identities (any necessary identities will be provided).
- You should also know the definitions of the arcsine and arctangent functions. These are also called the inverse sine and inverse tangent, but be careful: $\arcsin x$ is not the same as $\frac{1}{\sin x}$ and $\arctan x$ is not the same as $\frac{1}{\tan x}$. Rather, they are the functions that reverse sine and tangent: For example, $\arcsin(1) = \frac{\pi}{2}$ because $\sin(\frac{\pi}{2}) = 1$.

Important values of the arcsine and arctangent functions will be provided to you on the exam (see the practice midterm’s second page).
- You should be able to graph lines and recognize the graphs of other simple functions, as you learned in precalculus.
- You should be able to take limits of simple functions, including limits as $x \to \infty$.
- You should be able to apply L’Hôpital’s Rule in simple situations, namely for limits where substitution yields $0/0$ or $\infty/\infty$. If you do not know how to use L’Hôpital’s Rule, the steps are simple for the $0/0$ and $\infty/\infty$ cases, and outlined in Section 4.7.
• You should also know that $0/0$, $\infty/\infty$, $0 \cdot \infty$, and $\infty - \infty$ are indeterminate forms—these formal quantities cannot be assigned a single sensible value in $[-\infty, \infty]$ (so when they are encountered in limits, substitution is inadequate). ($1^{\infty}$ is also an indeterminate form, but it will definitely not appear on the midterm.)

• You should be able to take derivatives and evaluate simple antiderivatives (Table entries 1–7 under heading I in the textbook’s integration table). You should be able to use $u$-substitution or table entries for more complicated integrals. Table entries past I.7 will be provided to you if needed.

• While integration by parts is part of the Math 20 syllabus, you will not be required to use integration by parts to evaluate integrals on any exam in Math 21.

Limit comparison, growth/decay rates

• You should know that in order to determine if $f(x) \prec g(x)$, $f(x) \asymp g(x)$, or $f(x) \succ g(x)$ (as $x \to \infty$) one evaluates the limit

$$\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right|$$

(provided this limit exists). If this limit is 0, then $f \prec g$; if this limit is $+\infty$, then $f \succ g$; and if this limit is a positive finite constant, then $f \asymp g$.

• You should be able to decide if $f(x) \prec g(x)$, $f(x) \asymp g(x)$, or $f(x) \succ g(x)$ (as $x \to \infty$) based either on taking the limit as above or by remembering where the functions $f$ and $g$ fall on the growth/decay spectrum or “number line of functions.”

• You should know some basic rules for using $\prec, \asymp, \succ$, including but not limited to:
  
  – For any constant $c \neq 0$, $f(x) \asymp cf(x)$ (as $x \to \infty$),
  – If $f(x) \succ g(x)$, then $f(x) + g(x) \asymp f(x)$ (as $x \to \infty$)—“the law of the dominant term,”
  – If $f(x) \prec g(x)$, then $\frac{1}{f(x)} \succ \frac{1}{g(x)}$ (as $x \to \infty$),
  – If $f(x) \asymp g(x)$, then $\frac{1}{f(x)} \asymp \frac{1}{g(x)}$ (as $x \to \infty$),
  – If $f(x) \prec g(x)$ and $p > 0$, then $f(x)^p \prec g(x)^p$ (as $x \to \infty$),

provided all relevant limits exist.

• You should know that if $f(x)$ and $g(x)$ are polynomials, then $f(x) \asymp g(x)$ (as $x \to \pm \infty$) if $\deg f(x) = \deg g(x)$, and that $f(x) \prec g(x)$ (as $x \to \pm \infty$) if $\deg f(x) < \deg g(x)$.

• You should know that if $f(x) = A^x$ and $g(x) = B^x$, then $f(x) \asymp g(x)$ (as $x \to \pm \infty$) only if $A = B$ and $f(x) \prec g(x)$ (as $x \to \pm \infty$) only if $A < B$.

• You should know that $\ln x \prec x^p$ for any $p > 0$ (power/polynomial growth dominates logarithmic growth) and that $a^x \succ x^p$ for any $a > 1$ and any $p > 0$ (exponential growth dominates power/polynomial growth).
Evaluating improper integrals

- You should know that if an improper integral is simple—it has a single issue and that issue occurs at and endpoint—the first step is to rewrite the improper integral as a limit.

- If the definite integral inside the limit requires \( u \)-substitution, you may either (1) apply the substitution rule for definite integrals OR (2) do the indefinite integral first and then evaluate. What I mean:

\[
\begin{align*}
(1) \quad \int_0^\infty xe^{-x^2} \, dx &= \lim_{b \to \infty} \int_0^b xe^{-x^2} \, dx = \lim_{b \to \infty} \frac{1}{2} \int_0^{b^2} e^{-u} \, du = \frac{1}{2} \lim_{b \to \infty} \left[ -e^{-u} \right]_0^{b^2} \\
(2) \quad \int_0^\infty xe^{-x^2} \, dx &= \lim_{b \to \infty} \int_0^b xe^{-x^2} \, dx = (*) = \frac{1}{2} \lim_{b \to \infty} \left[ -e^{-x^2} \right]_0^b \text{ where } (*) \text{ is the side computation } \int xe^{-x^2} \, dx = -\frac{1}{2}e^{-x^2} + C
\end{align*}
\]

(In the integral above we used \( u = x^2 \) and \( du = 2x \, dx \).)

- You should be able to identify why an improper integral is improper and, if the improper integral is not simple, you should be able to write it as a sum of simple improper integrals by breaking up the interval of integration.

- You should know the following improper integrals by heart:

  a. \( \int_1^\infty \frac{1}{x^p} \, dx \) converges to \( \frac{1}{p - 1} \) if \( p > 1 \), and diverges if \( p \leq 1 \).
  b. \( \int_0^1 \frac{1}{x^p} \, dx \) converges to \( \frac{1}{1 - p} \) if \( p < 1 \), and diverges if \( p \geq 1 \).
  c. \( \int_0^\infty r^{-x} \, dx \) converges to \( \frac{1}{\ln r} \) if \( r > 1 \), and diverges if \( 0 < r \leq 1 \).
  d. \( \int_0^1 \ln x \, dx \) converges to \( -1 \).

Generalities about convergence of improper integrals

- You should know that if \( \int_a^b f(x) \, dx \) converges, then so does \( \int_c^d f(x) \, dx \) as long as no new issues are introduced changing the bounds from \( a, b \) to \( c, d \).

- You should know that if you have

  \[
  \text{complex improper integral} = \text{sum of simple improper integrals}
  \]

then if any of the simple improper integrals diverge, the integral on the left also diverges. The integral on the left converges only if all the integrals on the right converge.
Limit comparison for improper integrals with upper limit $\infty$

- You should know that if $f(x)$ and $g(x)$ are both “nice” (positive, defined, continuous, no asymptotes) on $[a, \infty)$ and $f(x) \asymp g(x)$ (as $x \to \infty$), then the integrals
  \[ \int_a^\infty f(x) \, dx \quad \text{and} \quad \int_a^\infty g(x) \, dx \]
either both converge or both diverge.

- You should know that if $f(x)$ and $g(x)$ are both nice on $[a, \infty)$ and $f(x) \prec g(x)$ (as $x \to \infty$), then
  \[ \int_a^\infty g(x) \, dx \]
  converges, so does \[ \int_a^\infty f(x) \, dx \]
  and \[ \int_a^\infty f(x) \, dx \]
  diverges, so does \[ \int_a^\infty g(x) \, dx \]

- You should know that the facts above only apply to integrals of the form $\int_a^\infty f$ and $\int_a^\infty g$ where $f$ and $g$ are “nice enough.” (No Type II improper integrals, and the upper endpoint must be $\infty$.)

- One common application is the following: If $P(x)$ and $Q(x)$ are polynomials and $Q(x)$ is never equal to zero on $[a, \infty)$, then
  \[ \int_a^\infty \frac{P(x)}{Q(x)} \, dx \]
  converges only if $\deg Q(x) - \deg P(x) > 1$, and diverges otherwise. How? Essentially because the integrand is $\asymp 1/x^p$ where $p = \deg Q - \deg P$ and we know that the improper integral (a.) converges if $p > 1$ and diverges otherwise (there are some small details, but this is the main idea).

- A common pitfall to avoid: Suppose you want to show that $\int_a^\infty \frac{f(x)}{g(x)} \, dx$ converges.
  It is NOT ENOUGH to show that $f(x) \prec g(x)$. This only shows that the integrand decays, but for an improper integral $\int_a^\infty h(x) \, dx$ to converge, $h(x)$ must decay quickly enough.
  For example, if $f(x) = 1$ and $g(x) = x$, then of course $f(x) \prec g(x)$, but $\int_1^\infty \frac{1}{x} \, dx$ diverges.

Direct comparison for integrals

- If $0 \leq f(x) \leq g(x)$ for $a < x < b$, then
  \[ 0 \leq \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx \]
  This is true both for proper integrals and improper integrals (of any type).
• Therefore, if you know that $0 \leq f(x) \leq g(x)$ for $a < x < b$:

\[
\begin{align*}
\text{if } & \int_a^b g(x) \, dx \text{ converges, so does } \int_a^b f(x) \, dx \\
\text{if } & \int_a^b f(x) \, dx \text{ diverges, so does } \int_a^b g(x) \, dx
\end{align*}
\]

• You will not need to know any overly tricky inequality manipulations for this class. Any improper integral whose convergence behavior requires direct comparison will be amenable to one of the following approaches:

  – The “problem term / bystander term” approach as outlined in notes on Piazza.
  – “An extra nonnegative term in the denominator”: If $h(x) \geq 0$ when $a < x < b$, then

\[
0 \leq \frac{f(x)}{g(x) + h(x)} \leq \frac{f(x)}{g(x)}
\]

So if you know that $\int_a^b \frac{f}{g}$ converges, $\int_a^b \frac{f}{g + h}$ does too.

  – A hint will be provided with an inequality that is useful for the problem.

• Useful inequality facts:

  – If $f(x)$ is increasing on $[a, b]$, then $f(a) \leq f(x) \leq f(b)$ for $a \leq x \leq b$. If $f(x)$ is decreasing on $[a, b]$, then $f(a) \geq f(x) \geq f(b)$ for $a \leq x \leq b$.
  – $a^x$ and $a^{-x}$ are always positive (provided $a > 0$).
  – $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$ for all real numbers $x$.

**Geometric sums and series**

• You should know the definition of a geometric sequence (an infinite list of numbers where the ratio of successive terms is constant), a geometric sum (a finite sum of consecutive terms in a geometric sequence), and a geometric series (an infinite sum of consecutive terms in a geometric sequence).

• You should know the formula for the geometric sum

\[
\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1} = \frac{1 - r^{n+1}}{1 - r}
\]

and the resulting formula for the geometric series,

\[
\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}
\]

which is valid when $-1 < r < 1$ (otherwise the geometric series diverges).