Failure to follow the instructions below is a breach of the Stanford Honor Code:

- You may not use or consult any book or notes during the exam.*
- You may not use a calculator or the calculator function on any electronic device during the exam.*
- You may not access any internet-capable electronic device during the exam,* including smartphones and smartwatches, for any reason. These devices must be switched to “airplane mode” and disconnected from all wireless networks (both cellular and wifi) during the exam*.
- You must sit in your assigned seat.
- You may not communicate with anyone other than the course staff during the exam,* or look at anyone else’s solutions.

*“During the exam” is defined as: After you start the exam, and before you turn in the exam and leave the testing site.

- You have 90 minutes to complete this exam. If the course staff must ask you to stop writing or to turn in your exam more than once after time is called, you may receive a score of zero.
- After you have turned in your exam, you may not discuss the contents of this exam with ANYONE other than the course staff until 9:00 PM tonight.
- If you need to make a phone call during the exam, ask a proctor.

I understand and accept these instructions. All smart devices on my person are in airplane mode and disconnected from all wireless networks.

Signature: _______________________________________________________

Remember to show your work and justify your answer if required (additional tips are on the next page). Present all solutions in as organized a manner as possible. GOOD LUCK!
Here are some tips:

- If you have time, it’s always a good idea to check your work when possible.
- If you get the wrong answer but show your work, you have a better chance of receiving partial credit.
- **DO NOT** attempt to estimate any of your answers as decimals. For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is exact.
- The very last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

### Integral table entries you may need

\[
\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C
\]
\[
\int \frac{du}{(u - r)(u - s)} = \frac{1}{r - s} \ln \left| \frac{u - r}{u - s} \right| + C
\]
\[
\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C
\]
\[
\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \left( \frac{u}{a} \right) + C
\]

In the entries above, $a, r, s$ are constants such that $a \neq 0$ and $r \neq s$.

### Values of arcsine and arctangent

The table below gives important values of the arcsine and arctangent functions:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{1}{\sqrt{3}}$</th>
<th>$\frac{\sqrt{3}}{2}$</th>
<th>1</th>
<th>$\sqrt{3}$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>arcsin $x$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/3$</td>
<td>$\pi/4$</td>
<td>$\pi/2$</td>
<td>undef.</td>
<td>undef.</td>
</tr>
<tr>
<td>arctan $x$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/3$</td>
<td>$\pi/4$</td>
<td>$\pi/2$</td>
<td>$\pi/3$</td>
<td>$\pi/2$</td>
</tr>
</tbody>
</table>

For negative values: $\arcsin(-x) = -\arcsin(x)$ and $\arctan(-x) = -\arctan(x)$.
Blank entries in the table are not “nice” multiples of $\pi$. 
Unless otherwise specified, in problems 1–6 you do not need to justify your answer.

1. In each space of the table below, write a $\text{C}$ if the integral in that column **CONVERGES** for the value of $p$ in that row. (If the integral diverges, you may leave it blank.)

<table>
<thead>
<tr>
<th></th>
<th>$\int_0^1 \frac{1}{x^p} , dx$</th>
<th>$\int_0^\infty \frac{1}{x^p} , dx$</th>
<th>$\int_1^\infty (1/x^p)^2 , dx$</th>
<th>$\int_0^1 \frac{1}{x^p} + 1 , dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $p = 1$</td>
<td></td>
<td>$\text{C}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b. $p = 1/3$</td>
<td>$\text{C}$</td>
<td></td>
<td></td>
<td>$\text{C}$</td>
</tr>
<tr>
<td>c. $p = 2$</td>
<td></td>
<td>$\text{C}$</td>
<td>$\text{C}$</td>
<td></td>
</tr>
<tr>
<td>d. $p = 4/5$</td>
<td>$\text{C}$</td>
<td>$\text{C}$</td>
<td>$\text{C}$</td>
<td>$\text{C}$</td>
</tr>
</tbody>
</table>

2. Complete the following statement: If $a > 0$, then $\int_0^\infty e^{-ax} \, dx$ is equal to...

If $a > 0$ then $\int_0^\infty e^{-ax} \, dx = \frac{1}{a}$ (per instructions, no justification was necessary).

3. Let $f(x)$ be a positive, continuous function.

In parts (a) and (b), circle the correct answer. Part (c) is at the top of the next page.

a. If $f(x)$ decays (i.e., if $\lim_{x \to \infty} f(x) = 0$), then $f(x)^2$ decays...

   FASTER  SLOWER  NEED MORE INFO ABOUT $f(x)$

b. If $f(x)$ grows without bound (i.e., if $\lim_{x \to \infty} f(x) = +\infty$), then $f(x)^2$ grows...

   FASTER  SLOWER  NEED MORE INFO ABOUT $f(x)$
c. We say that $f(x)$ is *bounded away from zero* if there is a constant $c > 0$ such that $f(x) \geq c$ for all $x \geq 1$. Justify the following statement:

If $f(x)$ is bounded away from zero, then $\int_1^\infty f(x) \, dx$ must diverge.

Your explanation doesn’t need to be overly long or technical, but it should be convincing.
If $f(x) \geq c$ for all $x \geq 1$ then by direct comparison of improper integrals,

$$\int_1^\infty f(x) \, dx \geq \int_1^\infty c \, dx$$

The integral on the right diverges. This can be seen either by evaluating it as a limit, or because it measures a rectangle of (positive) height $c$ and infinite length. Because the integral on the left diverges, the integral on the right diverges by direct comparison.
A pictorial explanation to this problem is given in the HW1 solutions, 3b.

4. Partario is attempting to determine whether the integral $\int_0^2 \frac{h(x)}{x} \, dx$ converges or diverges, where $y = h(x)$ is graphed on the next page.

Partario decides to use direct comparison to show that the integral *diverges*. His argument is below:

i. From the graph, we see that $0 \leq h(x) \leq 4$ whenever $0 \leq x \leq 2$.

ii. Therefore, $0 \leq \frac{h(x)}{x} \leq \frac{4}{x}$ whenever $0 < x \leq 2$.

iii. Integrating yields $0 \leq \int_0^2 \frac{h(x)}{x} \, dx \leq \int_0^2 \frac{4}{x} \, dx$.

iv. The integral $\int_0^2 \frac{4}{x} \, dx = 4 \int_0^2 \frac{1}{x} \, dx$ diverges...

v. ...so we may conclude that $\int_0^2 \frac{h(x)}{x} \, dx$ diverges too.

Partario’s argument is incorrect!

a. Which step(s) in his argument contains invalid reasoning?
*Circle the correct answer.*

i. ii. iii. iv. v.

Partario’s mistake is the following: If $A \leq B$ and $B$ diverges, you cannot conclude anything about $A$’s behavior. Every other step in his argument is correct.
b. Determine whether the integral \( \int_{0}^{2} \frac{h(x)}{x} \, dx \) converges or diverges, and give a correct argument; OR, state that there is not enough information about \( h(x) \) to do so.

The integral diverges:
When \( 0 \leq x \leq 1 \) (the left half of the interval graphed above) we have
\[
2 \leq h(x) \leq 4
\]

So \( \frac{2}{x} \leq \frac{h(x)}{x} \leq \frac{4}{x} \) when \( 0 < x \leq 1 \). Integrating from 0 to 1, we see that
\[
\int_{0}^{1} \frac{h(x)}{x} \, dx
\]
diverges (since \( \int_{0}^{1} \frac{2}{x} \, dx \) does). Since \( h(x)/x \) is defined (no asymptotes) when \( 1 \leq x \leq 2 \), we can change the upper endpoint of the integral without affecting convergence, so \( \int_{0}^{2} \frac{h(x)}{x} \, dx \) diverges too.

Another way to see this is using \( \int_{0}^{2} = \int_{0}^{1} + \int_{1}^{2} \). The first integral on the right diverges (by the argument above), and the second converges (being a proper integral) so the integral on the left diverges.

5. In (a–h) below, fill in the blank with the correct asymptotic relation \( \prec, \asymp, \text{ or } \succ \).

\textit{You do not need to justify your answers.}

\begin{align*}
a. \quad x^{1/5} & \succ x^{-5} & b. \quad x^{5} & \prec (2x^{2} + 4)^{3} & c. \quad x + e^{x} & \asymp x^{2} + e^{x} & d. \quad e^{x} & \prec (2x)^{2} \\
e. \quad e^{-x} & \succ \frac{x}{3^{x}} & f. \quad x + e^{-x} & \prec x^{2} + e^{-x} & g. \quad x^{0.001} & \asymp \ln(x) & h. \quad x & \asymp \frac{x^{5}}{(x^{2} + 1)^{2}}
\end{align*}
6. The *hyperbolic secant* function is defined by \( \text{sech}(x) = \frac{2}{e^x + e^{-x}} \) for all real numbers \( x \).

This function and its inverse, \( \text{arcsech}(x) \), (which is defined when \( 0 < x \leq 1 \)) are graphed below:

Above, \( y = \text{sech}(x) \) is the “bell-shaped” curve, while \( y = \text{arcsech}(x) \) is the curve with a vertical asymptote. Note that the curves in the right quadrant are each other’s reflections over the line \( y = x \). This is because \( \text{arcsech}(x) \) is the inverse of \( \text{sech}(x) \).

a. Briefly explain why \( \int_{0}^{\infty} \text{sech}(x) \, dx = \int_{0}^{\infty} \frac{2}{e^x + e^{-x}} \, dx \) converges.

The integrand is asymptotic to \( e^{-x} \) and \( \int_{0}^{\infty} e^{-x} \, dx \) converges, so \( \int_{0}^{\infty} \text{sech}(x) \, dx \) converges by limit comparison.

b. Suppose a friendly robot informs you that \( \int_{-\infty}^{\infty} \text{sech}(x) \, dx = 2\pi \). Using this information, evaluate \( \int_{0}^{1} \text{arcsech}(x) \, dx \). No justification required.

The regions measured by \( \int_{0}^{1} \text{arcsech}(x) \, dx \) and \( \int_{0}^{\infty} \text{sech}(x) \, dx \) are geometrically congruent (each can be obtained by reflecting the other over \( y = x \)) so they have the same area. Since \( \text{sech}(x) \) is even, by symmetry, \( \int_{0}^{\infty} \text{sech}(x) \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \text{sech}(x) \, dx \), and so \( \int_{0}^{1} \text{arcsech}(x) = (2\pi)/2 = \pi \).

*Note: The friendly robot was wrong: \( \int_{-\infty}^{\infty} \text{sech}(x) \) actually converges to \( \pi \), not \( 2\pi \). However, you were certainly not expected to notice this, so this problem was graded under the assumption that the robot was correct.*
In problems 7–9, evaluate the integral. Show your work. If you believe that the integral diverges, write DIVERGES as your final answer. Show all work and draw boxes around your final answers.

7. \[\int_{1}^{\infty} x^2 e^{-x^3} \, dx\]

Evaluating the indefinite integral first:

\[
\int x^2 e^{-x^3} \, dx = -\frac{1}{3} \int e^u \, du
\]

\[
= -\frac{e^u}{3} + C
\]

\[
= -\frac{e^{-x^3}}{3} + C
\]

where we use the substitution \( u = -x^3 \) and \( du = -3x^2 \, dx \) (so \( x^2 \, dx = -\frac{1}{3} \, du \)).

Now the improper integral:

\[
\int_{1}^{\infty} x^2 e^{-x^3} \, dx = \lim_{b \to \infty} \int_{1}^{b} x^2 e^{-x^3} \, dx
\]

\[
= \lim_{b \to \infty} \left[ -\frac{e^{-x^3}}{3} \right]_{1}^{b}
\]

\[
= \lim_{b \to \infty} \left[ -\frac{e^{-b^3}}{3} + \frac{e^{-1}}{3} \right]
\]

\[
= 0 + \frac{e^{-1}}{3}
\]

\[
= \frac{1}{3e}
\]

8. \[\int_{1}^{\infty} \frac{4}{x^2 + 4x} \, dx\] \quad \text{Hint:} \quad \frac{4}{x^2 + 4x} = \frac{1}{x} - \frac{1}{x + 4}.

Evaluating the indefinite integral first:

\[
\int \frac{4}{x^2 + 4x} \, dx = \int \left( \frac{1}{x} - \frac{1}{x + 4} \right) \, dx
\]

\[
= \int \frac{1}{x} \, dx - \int \frac{1}{x + 4} \, dx
\]

\[
= \ln |x| - \ln |x + 4| + C
\]

\[
= \ln \left| \frac{x}{x + 4} \right| + C
\]

Note: In the second line we break up \(\int (f + g) = \int f + \int g\). This is valid because the integrals are indefinite. In the last step we use \(\ln(A) - \ln(B) = \ln(A/B)\).
Now the improper integral:

\[
\frac{4}{x^2 + 4x} \, dx = \lim_{b \to \infty} \int_1^b \frac{4}{x^2 + 4x} \, dx
\]

\[
= \lim_{b \to \infty} \left[ \ln \left| \frac{x}{x + 4} \right| \right]_1^b
\]

\[
= \lim_{b \to \infty} \left[ \ln \left| \frac{b}{b + 4} \right| - \ln \left| \frac{1}{1 + 4} \right| \right]
\]

\[
= \ln |1| - \ln \left| \frac{1}{5} \right|
\]

\[
= 0 - \ln(1/5) = \ln(5)
\]

9. \( \int_{-3}^{2} \frac{1}{x^{7/6}} \, dx \)

DIVERGES. It must be broken up into \( \int_{-3}^{0} + \int_{0}^{2} \) and \( \int_{0}^{2} \frac{1}{x^{7/6}} \, dx \) diverges “by \( p \)-test.”

In problems 10–11, determine whether the integral CONVERGES or DIVERGES using the method of your choice. Be sure to justify your answer. Some tips:

- If you use limit or direct comparison, show enough work that we can follow your argument and understand how you came to your conclusion.
- You do not have to take limits step-by-step to verify asymptotic relations.
- You may use the convergence conditions for \( \int_{0}^{1} \frac{1}{x^{p}} \, dx \), \( \int_{1}^{\infty} \frac{1}{x^{p}} \, dx \), \( \int_{0}^{\infty} e^{-ax} \, dx \), and \( \int_{0}^{\infty} r^{-x} \, dx \) without further justification, but mention when you use them.

Draw boxes around your final answers.

10. \( \int_{0}^{\infty} \frac{x^{100}}{100^x} \, dx \)

There are a couple ways to approach this one using limit comparison. One is to remember that power growth is dominated by exponential growth, so \( x^{100} \prec 2^x \) for example. This gives us

\[
\frac{x^{100}}{100^x} \prec \frac{2^x}{100^x} = \frac{1}{50^x}
\]

and since \( \int_{0}^{\infty} 50^{-x} \, dx \) converges, our integral CONVERGES too. Other comparison are possible (and useful), such as \( \frac{x^{100}}{100^x} \prec \frac{1}{x^2} \).
The integrand has an asymptote at $x = 0$ and the integral extends to infinity, so we must first break it up as $\int_0^1 + \int_1^\infty$. Now we treat these integrals individually:

- The behavior of
  \[
  \int_1^\infty \frac{dx}{\sqrt{x^5 + 2x^3 + x}}
  \]
  can be quickly determined by limit comparison. The integrand is $\approx \frac{1}{x^{5/2}}$, so since $\int_1^\infty \frac{1}{x^{5/2}}$ converges, this part of the integral converges too.

- The integral
  \[
  \int_0^1 \frac{dx}{\sqrt{x^5 + 2x^3 + x}}
  \]
  requires using direct comparison.

  Using either the fact that $x^5 + 2x^3 \geq 0$ when $0 \leq x \leq 1$ or a problem-bystander argument, we obtain
  \[
  \frac{1}{\sqrt{x^5 + 2x^3 + x}} \leq \frac{1}{\sqrt{x}}
  \]

  Since $\int_0^1 \frac{1}{\sqrt{x}} \, dx$ converges, this part of the original integral converges too.

Since both parts of the integral converge, the whole integral CONVERGES.