Failure to follow the instructions below will constitute a breach of the Stanford Honor Code:

• You may not use a calculator or any notes or book during the exam.
• You may not access your phone or any other electronics during the exam for any reason.
• You must sit in your assigned seat.
• You may not communicate with anyone other than the course staff during the exam, or look at anyone else’s solutions.
• Additionally, you may not discuss the contents of this exam with ANYONE other than the course staff until 8:30 tonight.
• You have 75 minutes to complete this exam. If the course staff must ask you to stop writing or to turn in your exam more than once after time is called, you may receive a score of zero.

I understand and accept these instructions.

Signature: _______________________________________________________________

Remember to show your work and justify your answer if required (additional tips are on the next page). Present all solutions in as organized a manner as possible.

GOOD LUCK!
Here are some tips:

- If you have time, it’s always a good idea to check your work when possible.
- If you get the wrong answer but show your work, you have a better chance of receiving partial credit.
- DO NOT attempt to estimate any of your answers as decimals. For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is exact.
- The very last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

Integral table entries you may need

\[
\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right| + C
\]

\[
\int \frac{du}{(u-r)(u-s)} = \frac{1}{r-s} \ln \left| \frac{u-r}{u-s} \right| + C
\]

\[
\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C
\]

\[
\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \left( \frac{u}{a} \right) + C
\]

In the entries above, $a, r, s$ are constants such that $a \neq 0$ and $r \neq s$.

Values of arcsine and arctangent

The table below gives important values of the arcsine and arctangent functions:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{\sqrt{3}}{2}$</th>
<th>$\frac{\sqrt{3}}{3}$</th>
<th>1</th>
<th>$\sqrt{3}$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>arcsin $x$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/4$</td>
<td>$\pi/3$</td>
<td>$\pi/2$</td>
<td>undef.</td>
<td>undef.</td>
</tr>
<tr>
<td>arctan $x$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/3$</td>
<td>$\pi/4$</td>
<td>$\pi/3$</td>
<td>$\pi/2$</td>
<td></td>
</tr>
</tbody>
</table>

For negative values: arcsin($-x$) = arcsin($x$) and arctan($-x$) = -arctan($x$).
Blank entries in the table are not “nice” multiples of $\pi$. 
In problems 1–7 you do not need to justify your answer or show your work unless otherwise specified.

1. a. For which values of $p$ does $\int_0^1 \frac{1}{x^p} \, dx$ converge? For which values of $p$ does it diverge?
   
   Converges for $p < 1$, diverges for $p \geq 1$.

b. For which values of $p$ does $\int_1^\infty \frac{1}{x^p} \, dx$ converge? For which values of $p$ does it diverge?
   
   Converges for $p > 1$, diverges for $p \leq 1$.

c. For which values of $p$ does $\int_0^\infty \frac{1}{x^p} \, dx$ converge? For which values of $p$ does it diverge?
   
   Diverges for all values of $p$.

2. Fill in the blanks below with the appropriate asymptotic relation: $<, =, >$ (as $x \to \infty$).

   a. $\ln(\sqrt{x}) \quad \asymp \quad \ln(x^5)$

   b. $x(\ln x)^2 \quad < \quad x^{5/4}$

   c. $x^2 + e^x \quad \asymp \quad 1 + e^x$

   d. $\sqrt{x^3 + 1} \quad \asymp \quad x^{3/2}$

   e. $3^{-x} \quad < \quad e^{-x}$

   f. $x^2 + e^{-x} \quad \geq \quad 1 + e^{-x}$

   g. $(x^3 + 1)^7 \quad \geq \quad (x^5 + 1)^4$

3. Note that $\int_{-b}^{b} x \, dx = 0$ for any real number $b \geq 0$. It follows that

   $$\lim_{b \to \infty} \int_{-b}^{b} x \, dx = 0$$

   May we conclude that $\int_{-\infty}^{\infty} x \, dx = 0$? Explain why or why not.

   We may not:

   $$\int_{-\infty}^{\infty} x \, dx = \int_{-\infty}^{0} x \, dx + \int_{0}^{\infty} x \, dx$$

   and both integrals on the right diverge. Hence the integral on the left diverges as well.

4. a. Write down an example of a sequence that is convergent not bounded, or state that no such sequence exists.

   No such example exists: All convergent sequences are bounded.

b. Write down an example of a sequence that is bounded but not convergent, or state that no such sequence exists.

   $(-1)^n$ is bounded but not convergent.
c. Write down an example of a sequence that is convergent but not monotone, or state that no such sequence exists.

\((-1/2)^n\) is convergent but not monotone.

5. Consider the following sequence:

\[
1, \frac{1 + 4 + 4^2}{1 + 2 + 2^2 + 2^3 + 2^4}, \frac{1 + 4 + \cdots + 4^3}{1 + 2 + \cdots + 2^6}, \frac{1 + 4 + \cdots + 4^4}{1 + 2 + \cdots + 2^8}, \ldots
\]

That is, indexing from \(n = 0\) (so \(a_0 = 1\)), the \(n\)th term of the sequence is

\[
a_n = \frac{\sum_{k=0}^{n} 4^k}{\sum_{k=0}^{2n} 2^k} = \frac{\text{sum of } 4^k \text{ from } k = 0 \text{ to } k = n}{\text{sum of } 2^k \text{ from } k = 0 \text{ to } k = 2n}
\]

a. Find an explicit, closed-form formula for \(a_n\), the \(n\)th term of the sequence above, in terms of \(n\).

(“Closed-form” here means that your formula should not contain a \(\sum\) symbol or any use of \(\cdots\).)

Using the geometric sum formula:

\[
a_n = \frac{\sum_{k=0}^{n} 4^k}{\sum_{k=0}^{2n} 2^k} = \frac{4^{n+1} - 1}{3} \cdot \frac{1}{2^{2n+1} - 1}
\]

which is closed enough.

b. (Optional—EXTRA CREDIT) Does the sequence \(\{a_n\}\) as defined above converge or diverge?

If you believe it converges, compute its limit, showing your work.

It converges to \(2/3\): We have

\[
a_n = \frac{1}{3} \cdot \frac{4^{n+1} - 1}{2^{2n+1} - 1} = \frac{1}{3} \cdot \left[ \frac{4 \cdot 4^n - 1}{2 \cdot 4^n - 1} \right]
\]

as \(n \to \infty\), the term in the brackets above will tend to \(4/2 = 2\) (this can be verified using L'Hôpital’s rule, for example).

6. Let \(\{a_n\} = \{a_1, a_2, a_3, \ldots\}\) be a sequence, and let \(\{S_n\}\) be the associated sequence of partial sums:

\[
S_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n
\]

a. Which of the following statements below must be true about these sequences?

Circle all true statements.

i. If \(\{S_n\}\) converges, then \(\{a_n\}\) must also converge.

ii. If \(\{a_n\}\) converges to zero, then \(\{S_n\}\) must converge (not necessarily to zero).

iii. If \(\{a_n\}\) diverges, then \(\{S_n\}\) must also diverge.

iv. If \(\{a_n\}\) is monotone, then \(\{S_n\}\) must also be monotone.
v. If the terms of \( \{a_n\} \) are all positive, then \( \{S_n\} \) must be increasing.

vi. None of the above statements are true.

b. Suppose that \( S_n = \frac{n-1}{n} \). Does the infinite series \( \sum_{k=1}^{\infty} a_k \) converge or diverge?

Remember, the value of the infinite series is equal to the limit of the partial sums, so in this case, the series would converge to \( \lim \left( \frac{n-1}{n} \right) = 1 \).

7. Let \( f(x) \) be a function defined on \([a, \infty)\) where \( a \) is an integer.

a. The integral test states that under certain conditions, the improper integral \( \int_{a}^{\infty} f(x) \, dx \) and the infinite series \( \sum_{k=a}^{\infty} f(k) \) will either both converge or both diverge. What conditions are required to apply the integral test?

\( f \) must be positive, decreasing, and continuous.

b. In the case where the integral test applies—with all conditions from (a) holding on the whole interval \([a, \infty)\)—and the integral and series both converge, which of the following will be true about their values?

i. The value of the series is always \( \geq \) the value of the integral.

ii. The value of the series is always \( \leq \) the value of the integral.

iii. The value of the series is always \( = \) the value of the integral.

iv. None of the above.
In Problems 8–10, evaluate the improper integral.
Note: These problems were slightly different depending on which form you had. Both versions are given here.

8A. \( \int_0^\infty 2^{-x} \, dx \)

\[
\int_0^\infty 2^{-x} \, dx = \lim_{b \to \infty} \int_0^b 2^{-x} \, dx \\
= \lim_{b \to \infty} \int_0^b (1/2)^x \, dx \\
= \lim_{b \to \infty} \left[ \frac{(1/2)^x}{\ln(1/2)} \right]_0^b \\
= \lim_{b \to \infty} \left[ \frac{(1/2)^b}{\ln(1/2)} - \frac{(1/2)^0}{\ln(1/2)} \right] \\
= 0 - \frac{1}{\ln(1/2)} \\
= \frac{1}{\ln(2)}
\]

8B. \( \int_{-\infty}^3 2^{-x} \, dx \)

Proceeding as for 8A, we see that this integral will diverge, since \( 2^{-x} \to \infty \) as \( x \to -\infty \).

9A. \( \int_0^{\ln 5} \frac{e^x}{e^x - 1} \, dx \)

\[
\int_0^{\ln 5} \frac{e^x}{e^x - 1} \, dx = \lim_{a \to 0^+} \int_a^{\ln 5} \frac{e^x}{e^x - 1} \, dx \\
= \lim_{a \to 0^+} \int_{e^a - 1}^{e^{\ln 5} - 1} \frac{du}{u} \quad u = e^x - 1 \quad du = e^x \, dx \\
= \lim_{a \to 0^+} \int_{e^a - 1}^4 \frac{du}{u} \\
= \lim_{a \to 0^+} [\ln(u)]_{e^a - 1}^4 \\
= \lim_{a \to 0^+} [\ln(4) - \ln(e^a - 1)] \\
= \ln(4) - \ln(0)) \\
\text{Integral diverges.}
\]

9B. \( \int_0^{\ln 5} \frac{e^x}{\sqrt{e^x - 1}} \, dx \)

Same \( u \)-substitution as for 9A, but since \( \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C \), at the end we have \( 2\sqrt{4} - 2\sqrt{0} = 4 \), so this integral converges.
10A. \( \int_{-\infty}^{\infty} \frac{dx}{4x^2 + 25} \)

The integrand is even so it is enough to integrate \( \int_{0}^{\infty} \) and then double the answer. We will use the table entry provided:

\[
\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C
\]

with \( u = 2x, \ du = 2 \, dx \), and \( a = 5 \) (a common mistake is to forget to compute \( du \) here and compensate for its difference with \( dx \)). Just evaluating the indefinite integral now,

\[
\int \frac{dx}{4x^2 + 25} = \frac{1}{2} \int \frac{du}{u^2 + a^2} = \frac{1}{2a} \arctan \left( \frac{u}{a} \right) + C = \frac{1}{10} \arctan \left( \frac{2x}{5} \right) + C
\]

For the improper integral,

\[
\int_{0}^{\infty} \frac{dx}{4x^2 + 25} = \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{4x^2 + 25} = \lim_{b \to \infty} \left[ \frac{1}{10} \arctan \left( \frac{2x}{5} \right) \right]_{0}^{b} = \lim_{b \to \infty} \left[ \frac{1}{10} \arctan \left( \frac{2b}{5} \right) - \frac{1}{10} \arctan(0) \right] = \frac{1}{10} \cdot \frac{\pi}{2} - \frac{1}{10} \cdot 0 = \frac{\pi}{20}
\]

So \( \int_{-\infty}^{\infty} \frac{dx}{4x^2 + 25} = \frac{\pi}{10} \).

10B. \( \int_{-\infty}^{\infty} \frac{dx}{9x^2 + 4} \)

Very similar to 10A only with different constants. The integral in this case converges to \( \frac{\pi}{6} \).
In Problems 11–13, determine if the improper integral converges or diverges using the method of your choice.

Note: These problems were slightly different depending on which form you had. Both versions are given here.

11A. \[ \int_\pi^\infty \frac{1 + (\sin x)^2}{x^{4/3}} \, dx \]

We have \( 0 \leq (\sin x)^2 \leq 1 \) so the numerator is bounded between 1 and 2. Thus
\[
\frac{1}{x^{4/3}} \leq \frac{1 + (\sin x)^2}{x^{4/3}} \leq \frac{2}{x^{4/3}}
\]
Integrating from \( \pi \) to \( \infty \), we have
\[
\int_\pi^\infty \frac{1}{x^{4/3}} \, dx \leq \int_\pi^\infty \frac{1 + (\sin x)^2}{x^{4/3}} \, dx \leq 2 \int_\pi^\infty \frac{1}{x^{4/3}} \, dx
\]
The integrals on either side converge (by the \( p \)-test; note that changing the lower endpoint from 1 to \( \pi \) does not change the conditions for convergence), so the integral in the middle converges as well.

11B. \[ \int_\pi^\infty \frac{1 + (\sin x)^2}{x^{3/4}} \, dx \]

The same comparison argument as above works for this integral as well, except now the integrals on the sides diverge by \( p \)-test, so the integral diverges.

12A. \[ \int_3^\infty \sqrt{\frac{2x + 1}{x^3 + x^2 + x + 1}} \, dx \]

The integrand is asymptotic to \( \sqrt{\frac{2x}{x^3}} = \sqrt{\frac{2}{x^2}} = \frac{1}{x} \), so the integral will diverge, since \( \int_3^\infty \frac{1}{x} \, dx \) does by \( p \)-test.

12B. \[ \int_3^\infty \sqrt{\frac{2x + 1}{x^4 + x^3 + x + 1}} \, dx \]

The integrand is now asymptotic to \( \sqrt{\frac{2x}{x^4}} = \sqrt{\frac{1}{x^3}} = \frac{1}{x^{3/2}} \), and \( \int_3^\infty \frac{1}{x^{3/2}} \, dx \) converges by \( p \)-test, so the integral in the problem converges as well.

13. \[ \int_0^1 \sqrt{\frac{x}{x^3 + x^2}} \, dx \quad (Same on both forms.) \]

Our integral can be simplified to \( \int_0^1 \frac{1}{\sqrt{x^2 + x}} \, dx \). Factoring the integrand yields
\[
\int_0^1 \frac{1}{\sqrt{x + 1}} \cdot \frac{1}{\sqrt{x}} \, dx
\]
And \( \frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{x + 1}} \leq 1 \) when \( 0 \leq x \leq 1 \). Thus,
\[
\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{x + 1}} \cdot \frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}
\]
and integrating yields
\[
\frac{1}{\sqrt{2}} \int_0^1 \frac{1}{\sqrt{x}} \, dx \leq \int_0^1 \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{x + 1}} \, dx \leq \int_0^1 \frac{1}{\sqrt{x}} \, dx
\]
The integrals on the sides converge, so so does the integral in the middle.