Failure to follow the instructions below will constitute a breach of the Stanford Honor Code:

- You may not use consult any book or notes during the exam.*
- You may not use a calculator or the calculator function on any electronic device during the exam.*
- You may not access any internet-enabled electronic device during the exam,* including smartphones, for any reason except to check the time. You may only check the time on your device if and only if you (1) Do not access any other information, and (2) Switched off all network/internet capabilities before the start of the exam (i.e., it is in airplane mode).
- You must sit in your assigned seat.
- You may not communicate with anyone other than the course staff during the exam,* or look at anyone else’s solutions.
- *“During the exam” is defined as: After you start the exam, and before you turn in the exam and leave the testing site.
- You have 80 minutes to complete this exam. If the course staff must ask you to stop writing or to turn in your exam more than once after time is called, you may receive a score of zero.
- You may not discuss the contents of this exam with ANYONE other than the course staff until 9:00 PM tonight.

I understand and accept these instructions.

Signature: ____________________________

Remember to show your work and justify your answer if required (additional tips are on the next page). Present all solutions in as organized a manner as possible. GOOD LUCK!
Here are some tips:

- If you have time, it’s always a good idea to check your work when possible.
- If you get the wrong answer but show your work, you have a better chance of receiving partial credit.
- DO NOT attempt to estimate any of your answers as decimals. For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is exact.
- The very last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

**Integral table entries you may need**

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C$$

$$\int \frac{du}{(u - r)(u - s)} = \frac{1}{r - s} \ln \left| \frac{u - r}{u - s} \right| + C$$

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C$$

$$\int \frac{-du}{\sqrt{a^2 - u^2}} = \arcsin \left( \frac{u}{a} \right) + C$$

In the entries above, $a, r, s$ are constants such that $a \neq 0$ and $r \neq s$.

**Values of arcsine and arctangent**

The table below gives important values of the arcsine and arctangent functions:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{1}{\sqrt{3}}$</th>
<th>$\frac{\sqrt{3}}{2}$</th>
<th>1</th>
<th>$\sqrt{3}$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>arcsin $x$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/3$</td>
<td>$\pi/4$</td>
<td>$\pi/2$</td>
<td>undef.</td>
<td>undef.</td>
</tr>
<tr>
<td>arctan $x$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/4$</td>
<td>$\pi/3$</td>
<td>$\pi/4$</td>
<td>$\pi/3$</td>
<td>$\pi/2$</td>
</tr>
</tbody>
</table>

For negative values: $\arcsin(-x) = -\arcsin(x)$ and $\arctan(-x) = -\arctan(x)$.

Blank entries in the table are not “nice” multiples of $\pi$. 
1. For which values of $p$ does the integral $\int_0^1 \frac{1}{x^p} \, dx$ converge? Describe all such values.

$p < 1$

2. For which values of $p$ does the integral $\int_1^\infty \frac{1}{x^p} \, dx$ converge? Describe all such values.

$p > 1$

3. In (a.–h.) fill in the blank with the correct asymptotic relation: $\asymp$, $\subset$, $<$, or $>$. (Explanations:

| (a) $x^5 \underset{\subset}{\asymp} (4x^3 + 1)^2$ | \[ b/c \quad (4x^3 + 1)^2 = 16x^6 + \ldots \quad \leftarrow \text{degree 6} \] |
| (b) $e^x \underset{\subset}{\asymp} 3^x$ | \[ b/c \quad e < 3 \] |
| (c) $e^{-x} \underset{\supset}{\asymp} e^{-x^2}$ | \[ b/c \quad \lim_{x \to \infty} \frac{e^{-x^2}}{e^{-x}} = e^{-(x^2 - x)} = e^{-\infty} = 0 \] |
| (d) $\ln(x) \underset{\subset}{\asymp} x^{1/1000}$ | \[ b/c \quad \ln(x) \asymp x^p \quad \text{for all } p > 0 \] |
| (e) $x \underset{\asymp}{\asymp} \sqrt{9x^2 - 25}$ | \[ b/c \quad \sqrt{9x^2 - 25} \asymp \sqrt{9x^2} = 3x \asymp x \] |
| (f) $\ln(x^5) \underset{\asymp}{\asymp} \ln(x^{1/5})$ | \[ b/c \quad \ln(x^5) = 5 \ln x \asymp \frac{1}{5} \ln x = \ln(x^{1/5}) \] |
| (g) $x(\ln x)^4 \underset{\subset}{\asymp} x^2$ | \[ b/c \quad \ln x \asymp x^{1/4} \]
\[ (\ln x)^4 \asymp x \]
\[ 3 \times (\ln x)^4 \asymp x^2. \] |
4. Above are graphed \( y = 2^x \) and \( y = \log_2(x) \). (Here, \( \log_2 \) is the base-2 logarithm: \( \log_2(2^x) = x \).)

We define \( A = \int_{-\infty}^{0} 2^x \, dx \) and \( B = \int_{0}^{1} \log_2(x) \, dx \).

a. In the graph at the top of the page, shade the regions corresponding to \( A \) and \( B \), labeling which region is which.

b. What is the value of the integral \( A \)? If you believe that the integral diverges, write DIVERGES. You do not need to show your work. Draw a box around your answer.

\[
\int 2^x \, dx = \frac{2^x}{\ln 2} + C \quad \text{Explanation: Integral formula (I.3)}
\]

so
\[
\int_{-\infty}^{0} 2^x \, dx = \lim_{a \to -\infty} \left[ \frac{2^0 - 2^a}{\ln 2} \right] = \left[ \frac{1}{\ln 2} \right] \quad \text{also equal to}
\]
\[
\int_{0}^{\infty} 2^{-x} \, dx \text{ by u-sub} \quad u = -x \quad du = -dx.
\]

\[
\int_{0}^{1} \log_2(x) \, dx
\]

4. What is the relationship between the values of \( A \) and \( B \)?

Circle the correct answer:

i. \( B = A \)   ii. \( B = 1/A \)   iii. \( B = -A \)   iv. \( B = -1/A \)   v. None of (i.–iv.).

\[\text{by symmetry, regions have same area, but } B \text{ is negative.} \]
5. Suppose that \( f(x) \) and \( g(x) \) are "nice" (defined, continuous, and positive) functions on \([0, \infty)\). Suppose furthermore that
\[
f(x) \prec g(x) \quad \text{as} \quad x \to \infty.
\]
Which of the following statements must be true, given the information above?
Circle all true statements.

\( \times \) If \( \int_0^\infty f(x) \, dx \) and \( \int_0^\infty g(x) \, dx \) both converge, then \( \int_0^\infty f(x) \, dx \leq \int_0^\infty g(x) \, dx \).

\( \times \) If \( \int_0^\infty f(x) \, dx \) converges, then \( \int_0^\infty g(x) \, dx \) also converges.

\( \text{i.} \) If \( \int_0^\infty g(x) \, dx \) converges, then \( \int_0^\infty f(x) \, dx \) also converges.

\( \text{iv.} \) If \( \int_0^\infty \frac{1}{f(x)} \, dx \) converges, then \( \int_0^\infty \frac{1}{g(x)} \, dx \) also converges.

\( \times \) If \( \int_0^\infty \frac{1}{g(x)} \, dx \) converges, then \( \int_0^\infty \frac{1}{f(x)} \, dx \) also converges.

\( \text{vi.} \) If \( \int_0^\infty f(x) \, dx \) converges, then \( \int_0^\infty f(x)^2 \, dx \) also converges.

\( \times \) None of the above.

Explanations

6. What is the value of the geometric sum
\[5^2 - 5^3 + 5^4 - 5^5 + \cdots + 5^{100}\]

Once you’ve applied the geometric sum formula you do not need to simplify your answer. Draw a box around your answer.

\[
\sum_{k=2}^{100} (-5)^k = \frac{(-5)^2 - (-5)^{101}}{1 - (-5)} = \boxed{\frac{5^2 + 5^{101}}{6}}
\]

\( \text{i.} \) Not necessarily: \( 3e^{-2x} \prec e^{-x} \) but \( \int_0^\infty 3e^{-2x} \, dx = 3 \cdot \frac{1}{2} = 1.5 \). While \( \int_0^\infty e^{-x} \, dx = 1 \).

\( \text{ii.} \) Not necessarily: \( e^{-x} \prec 1 \) and \( \int_0^\infty e^{-x} \, dx \) converges, while \( \int_0^\infty 1 \, dx = \infty \).

\( \text{iii.} \) Limit comparison!

\( \text{iv.} \) If \( f(x) \prec g(x) \), then \( \frac{f(x)}{g(x)} \prec \frac{1}{g(x)} \) so this is L. Comp.

\( \text{v.} \) No. Take \( f(x) = 1 \) and \( g(x) = e^x \).

\( \text{vi.} \) If \( \int f \) converges, then \( f < 1 \), so \( f^2 < f \); so \( \int f^2 \) converges.
7. Below are given two geometric series $P$ and $Q$. One of them converges and the other diverges. Circle the convergent series and then compute its value:

$$P = \frac{1}{1000} + \frac{3}{1000} + \frac{9}{1000} + \frac{27}{1000} + \cdots$$

$$Q = \frac{1000}{1} + \frac{1000}{3} + \frac{1000}{9} + \frac{1000}{27} + \cdots$$

Circle one of $P$ or $Q$ above, then write the value of the convergent series below and draw a box around your answer.

$$Q = \sum_{k=0}^{\infty} \frac{1000}{3^k} = 1000 \cdot \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k$$

$$= 1000 \cdot \frac{1}{1 - \frac{1}{3}}$$

$$= 1000 \cdot \frac{3}{2} = \boxed{1500}$$

8. Archimedes developed a method for computing the area under a parabola roughly 2000 years before the invention of integral calculus(!). Without getting into the details, he proved that the area under the parabola is equal to the infinite series

$$\left(\frac{T}{8}\right) + 2\left(\frac{T}{8^2}\right) + 4\left(\frac{T}{8^3}\right) + \cdots$$

where $T$ is the area of a triangle inscribed in the parabola (as in the diagram).

The series above is geometric. That is, it has the form $a + ar + ar^2 + ar^3 + \cdots$ for some values of $a$ and $r$. What are these values?

$$a = \frac{T}{8}$$

$$\frac{T}{8} + 2\left(\frac{\frac{T}{8}}{8}\right) + 4\left(\frac{\frac{T}{8^2}}{8}\right) + \cdots$$

$$r = \frac{2}{8} = \frac{1}{4}$$
In Problems 9–11, evaluate the improper integral. Show your work. If you believe the integral to diverge, write DIVERGES as your final answer. Draw boxes around your final answers.

9. $\int_{-1}^{4} 3x^{-2} \, dx$

(0) Find issues and split if necessary: Asymptote at $x=0$

\[ = \int_{-1}^{0} 3x^{-2} \, dx + \int_{0}^{4} 3x^{-2} \, dx \leftarrow \text{have to split so asymptote is at endpoints} \]

This integral = $3 \left[ \int_{0}^{1} \frac{1}{x^2} \, dx + \int_{1}^{4} \frac{1}{x^2} \, dx \right]$

and $\int_{0}^{1} \frac{1}{x^2} \, dx$ diverges, so our integral

\[ \boxed{\text{DIVERGES}} \]

10. $\int_{0}^{1} \frac{e^x}{\sqrt{e^x - 1}} \, dx$

(0) Find issues: asymptote at $x=0$ (no split!)

(1) Rewrite as limit of proper integrals:

\[
\left[ \lim_{a \to 0^+} \int_{a}^{1} \frac{e^x}{\sqrt{e^x - 1}} \, dx \right]
\]

\[ \downarrow u = e^x - 1, \, du = e^x \, dx \]

\[
= \lim_{a \to 0^+} \int_{e^a - 1}^{e^1 - 1} \frac{du}{\sqrt{u}}
\]

\[
= \lim_{a \to 0^+} \left[ 2\sqrt{u} \right]_{e^a - 1}^{e^1 - 1}
\]

\[
= \lim_{a \to 0^+} \left[ 2\sqrt{e - 1} - 2\sqrt{e^a - 1} \right]
\]

(3) Evaluate limit

\[
= \sqrt{2e - 1} \rightarrow 0 \text{ as } a \to 0^+
\]

Also could use indefinite $\int$:

\[
= \int \frac{e^x}{\sqrt{e^x - 1}} \, dx
\]

\[= \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C\]

\[= 2\sqrt{e^x - 1} + C\]
11. Evaluate the integral \( \int_0^\infty \frac{3}{x^2 + 9x + 18} \, dx \).

Hint: \( \frac{3}{x^2 + 9x + 18} = \frac{1}{x + 3} - \frac{1}{x + 6} \).

(0) Find issues: upper endpoint is \( \infty \), no split.

(1) Rewrite as limit:

\[
\lim_{b \to \infty} \int_0^b \frac{3}{x^2 + 9x + 18} \, dx \quad \text{by hint}
\]

\[
= \lim_{b \to \infty} \left[ \int_0^b \left( \frac{1}{x + 3} - \frac{1}{x + 6} \right) \, dx \right]
\]

\[
= \lim_{b \to \infty} \left[ \int_0^b \frac{dx}{x + 3} - \int_0^b \frac{dx}{x + 6} \right]
\]

\[
= \lim_{b \to \infty} \left[ \ln|x + 3| - \ln|x + 6| \right]_0^b \quad \text{by log rules}
\]

\[
= \lim_{b \to \infty} \left[ \ln \left| \frac{x + 3}{x + 6} \right| \right]_0^b
\]

\[
= \lim_{b \to \infty} \left[ \ln \left| \frac{b + 3}{b + 6} \right| - \ln \left| \frac{b}{b + 6} \right| \right]
\]

\[
= \ln(4) - \ln \left( \frac{1}{2} \right) \quad \text{by log rules}
\]

\[
= 0 + \left| \ln 2 \right|
\]
In Problems 12–14, determine whether the integral CONVERGES or DIVERGES using the method of your choice. Justify your answer!

- If you evaluate the integral, show all steps.
- If you use limit or direct comparison, outline your argument well enough that we can understand how you came to your conclusion.
- You may use the convergence conditions for \( \int_0^1 \frac{1}{x^p} \, dx \), \( \int_1^\infty \frac{1}{x^p} \, dx \), and \( \int_0^\infty r^{-2} \, dr \) without justification, but mention those conditions if you use them.

Draw boxes around your final answers.

12. \( \int_\pi^\infty \frac{5 + 2 \cos \theta}{\theta^3} \, d\theta \)

Direct comparison: \(-1 \leq \cos \theta \leq 1\), so

\[
5 - 2 \leq 5 + 2 \cos \theta \leq 5 + 2
\]

Thus,

\[
\frac{5}{\theta^2} \leq \frac{5 + 2 \cos \theta}{\theta^3} \leq \frac{7}{\theta^3}
\]

Thus, \( \int_\pi^\infty \frac{1}{\theta^3} \, d\theta \) converges, so our integral \( \int_\pi^\infty \frac{5 + 2 \cos \theta}{\theta^3} \, d\theta \) \( \text{CONVERGES} \)

13. \( \int_1^\infty \frac{5x^3 + 10x + 4}{18x^5 + 9x^3 + x + 1} \, dx \)

Limit comparison

Integrand \( \frac{\sqrt{5x^3}}{18x^5} \cdot \frac{1}{x^2} = \frac{1}{x} \), so since \( \int_1^\infty \frac{1}{x} \, dx \) diverges, our integral \( \int_1^\infty \frac{5x^3 + 10x + 4}{18x^5 + 9x^3 + x + 1} \, dx \) \( \text{DIVERGES} \)
14. \( \int_0^{\infty} \frac{dx}{4x^5 + \sqrt{x}} \)

- **Identify issues:**
  - upper endpt = \( \infty \)
  - lower endpt = 0 and there's an asymptote at \( x = 0 \).

**Split:** \( S_0 = S_0' + S_1' \)

**(a)\** \( S_0' = \int_0^{\infty} \frac{dx}{4x^5 + \sqrt{x}} \)

**Direct comparison:**
for \( x \geq 0 \)
\[ \sqrt{x} \leq 4x^5 + \sqrt{x} \]
so \( 0 \leq \frac{1}{4x^5 + \sqrt{x}} \leq \frac{1}{\sqrt{x}} \)
\( \downarrow S_0' \)

\( 0 \leq \text{our } S_0' \leq \text{converges} \)

**So (A) converges.**

(Could also use problem-bystander:
\[ \frac{1}{4x^5 + \sqrt{x}} = \left( \frac{1}{\sqrt{x^9} + 1} \right) \])

**(b)\** \( S_1' = \int_1^{\infty} \frac{dx}{4x^5 + \sqrt{x}} \)

**Limit comparison:**
\[ \frac{1}{4x^5 + \sqrt{x}} \leq \frac{1}{x^5} \text{, so since} \]
\( \int_1^{\infty} \frac{1}{x^5} \text{ } dx \text{ converges,} \)

**B converges.**

Since \( S_0 = (\text{A}) + (\text{B}) \) both converge,
our integral

**CONVERGES**