Midterm 2 Expectations

Midterm 2 will be 80 minutes long and will cover the following topics: Sequences and their properties, infinite series, convergence tests, power series, and basic power series manipulations. This corresponds roughly to sections 9.1–9.5 of the textbook, with some additions and subtractions—all testable material is described below.

Note that while geometric series were covered in Midterm 1, these are central to the material covered by Midterm 2 as well, so they may still appear. That being said, you will not be tested on improper integrals directly, but they may still play a role via e.g. the integral test.

Prerequisites

Same as for Midterm 1.

Asymptotic comparison, growth/decay rates

- Asymptotic comparison is still important for Midterm 2. There won’t be any questions specifically about it (such as Problem 2 on Practice Midterm 1C) but it is particularly useful for determining series convergence (via the limit comparison test).

- In addition to the asymptotic comparison facts from Midterm 1, you should know that $n!$ grows superexponentially: $n! \asymp b^n$ for all $b > 0$ and that $n! \ll n^n$.

- Stirling’s approximation $n! \sim n^{n+1/2}e^{-n}$ will be provided, should you need it.

Sequences and series

- You should know what a sequence is (an infinitely long list of numbers).

- You should know the definitions of convergent and divergent sequences, bounded sequences, and monotone sequences.

- You should know the relationships between the sequence properties:
  - If $\{a_n\}$ converges, then $\{a_n\}$ is bounded.
  - If $\{a_n\}$ is bounded and monotone, then $\{a_n\}$ converges.

You should know that the converse statements are not true and you should be prepared to provide counterexamples: A bounded non-convergent sequence, and a convergent non-monotone sequence.

Infinite series and convergence tests

- You should be familiar with the following special series and their convergence conditions:

  - Geometric series: $\sum_{n=0}^{\infty} r^n$, which converges to $\frac{1}{1 - r}$ if $|r| < 1$ and diverges if $|r| \geq 1$. 
- **p-series**: \( \sum_{n=1}^{\infty} \frac{1}{n^p} \), which converges if \( p > 1 \) and diverges if \( p \leq 1 \).

- Harmonic series: \( \sum_{n=1}^{\infty} \frac{1}{n} \), the \( p = 1 \) case of the \( p \)-series; this diverges.

- You should know the difference between the **terms** of a series and the **partial sums** of a series (e.g., Problem 6 on practice Midterm 1C). You should know that the value of a convergent series is equal to the limit of its partial sums.

- You should know the basic properties of sequences outlined in Theorem 9.2 of your textbook.

  A word of caution about Theorem 9.2.1: If either \( \sum a_n \) or \( \sum b_n \) diverges, it is not necessarily true that \( \sum (a_n + b_n) = \sum a_n + \sum b_n \). See for example Problem 3 on Practice Midterm 2B.

- You should be able to apply the following tests for series convergence/divergence:
  - Divergence test (“If \( \{a_n\} \) does not tend to zero, then \( \sum a_n \) diverges.”)
  - Integral test
  - Direct comparison test
  - Limit comparison test
  - Ratio test
  - Alternating series test
  - Absolute convergence test (“If \( \sum |a_n| \) converges, then so does \( \sum a_n \).”)

- You should know the difference between **absolute** and **conditional** convergence, and you should know at least one example of a series that converges conditionally (but not absolutely).

- You should be aware of the following common pitfalls for series convergence tests:
  - You cannot apply series convergence tests to test the convergence/divergence of **sequences**. For **sequence convergence**, one must usually only evaluate a simple limit.
  - The divergence test cannot prove that a series converges: If \( \lim_{n \to \infty} (a_n) = 0 \), then \( \sum a_n \) might converge or it might diverge.
  - To apply direct or limit comparison, the terms of the sequence must be (at least eventually) all \( \geq 0 \). For example, to show that \( \sum \frac{\sin n}{n^2} \) converges, you cannot apply direct comparison directly, because \( \sin n \) will be negative for infinitely many values of \( n \). (For this series, use the absolute convergence test and then direct comparison with \( 0 \leq |\sin n| \leq 1 \).)

- You should know the following basic strategies for testing series convergence:
  - If \( \sum a_n \) is alternating, check \( \sum |a_n| \) for convergence first. If \( \sum |a_n| \) converges, then \( \sum a_n \) converges absolutely. If \( \sum |a_n| \) diverges, \( \sum a_n \) could still converge conditionally—use the alternating series test to check.
  - Series whose terms are algebraic (do not include \( n \) in exponents other than \((-1)^n\) and do not contain factorials) can usually be tested successfully using **limit comparison** with a \( p \)-series. (The ratio test will be inconclusive for such series, with \( L = 1 \).)
Series whose terms include $n$ in exponents or contain factorials can usually be tested successfully using the **ratio test**.

Series with sine or cosine: (a) If the series has $\sin n$ or $\cos n$, use **direct comparison** (possibly preceded by the **absolute convergence test**) using the fact that $-1 \leq \sin n, \cos n \leq 1$. (b) If the series has a sine or cosine with $\pi$ in its input (like $\sin(n\pi/2)$ for example), then the sine or cosine term (i) might always be equal to one of $1, 0, -1$, or (ii) might cause the series to become alternating. You have to test values of $n$ to diagnose the series’ behavior and then proceed accordingly.

Series with logarithms: (a) Either use **limit comparison** based on the fact that $\ln n \prec n^p$ for any positive $p$ (you will have to choose a value of $p$ to work with); or if all else fails, (b) the **integral test** where evaluation of the integral will probably require the substitution $u = \ln x$. Log rules, which have never stopped being important, may also play a role.

The bounds from the integral test and alternating series test will be provided for you on the exam, should you need them.

You should be able to apply the error bounds from the alternating series test, in principle, to the problem of estimating the value of a convergent infinite series to within a specified margin of error. (Problem 2b on Practice Midterm 2C.)

**Power series convergence**

- You should know the process for finding the interval of convergence for a power series:
  * Step 0: Find the center (this should be fairly obvious from how the power series is written).
  * Step 1: Apply the ratio test to the power series. You will obtain a function $L(x)$. The series converges absolutely when $L(x) < 1$, diverges when $L(x) > 1$, and the endpoints of the interval of convergence are located where $L(x) = 1$.
    (Note that if you are only asked to find the **radius** of convergence, you do not need to do Step 2: The radius of convergence is the distance from the center to either endpoint.)
  * Step 2: Test the endpoints by plugging them into the series and checking convergence as usual.
- You should know that in the interior of the IoC, convergence is **absolute**, but, at each of the endpoints, the power series may diverge, converge conditionally, or converge absolutely.
- You may use the following shortcut for testing endpoints: If the power series contains only even powers or only odd powers of $(x - a)$, then it has the same convergence behavior at both endpoints of its IoC. (So it suffices to test one of them.)
- At the endpoints the ratio test **will not work**, since the endpoints are those $x$ values for which $L(x) = 1$.

**Power series representations**

- You should know the following **four** power series representations / Taylor series expansions:
  1. $\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$ on $(-1, 1)$;
  2. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ on $(-\infty, \infty)$;
3. There are various (equivalent) ways to represent natural log as a power series:

a. \[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = \ln(1 + x) \text{ on } (-1, 1]; \] and/or

b. \[ \sum_{n=0}^{\infty} \frac{(-1)^n (x - 1)^n+1}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x - 1)^n}{n} = \ln(x) \text{ on } (0, 2]; \]

4. \[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan x \text{ on } [-1, 1]. \]

- You should be able to find new power series representations from old power series representations using elementary transformations.

For example, \( \sum_{n=0}^{\infty} 4^n x^{3n+1} = x \sum_{n=0}^{\infty} (4x^3)^n \) represents \( \frac{x}{1-4x^3} \) on its IoC.

- You should know that such transformations may change the interval of convergence.

- You will not be asked to use the non-elementary power series transformations of integration and differentiation on Midterm 2, though you should know that the power series for \( \ln(x + 1) \) and \( \arctan x \) were obtained by integrating power series representing \( \frac{1}{x+1} \) and \( \frac{1}{x^2+1} \), respectively.

- You should know that if a power series \( F(x) \) represents the function \( f(x) \) on an interval \( I \), then for any number \( c \) in the interval \( I \), \( F(c) \) converges to \( f(c) \). You should be able to use this fact to derive series representations of constants and evaluate some infinite series. For example:

* Since \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) represents \( e^x \) on \( (-\infty, \infty) \) (by formula (2) above), plugging in \( x = -2 \) shows that the series \( \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} \) converges to \( e^{-2} = \frac{1}{e^2} \).

* Evaluate \( \sum_{n=1}^{\infty} \frac{4^n}{5^n \cdot n} \):

\[
\sum_{n=1}^{\infty} \frac{4^n}{5^n \cdot n} = \sum_{n=1}^{\infty} \frac{(4/5)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-4/5)^n}{n} \\
= -\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (-4/5)^n}{n} = -\ln(1 - \frac{4}{5}) = \ln 5
\]

where in the last line we use formula (3a) from above (note that \( x = -\frac{4}{5} \) is in the IoC, \( (-1, 1] \)).

- You should know that if \( \sum c_n(x - a)^n \) represents \( f(x) \) on an interval centered at \( a \), then \( c_n \) is related to \( f^{(n)}(a) \) (the \( n \)th derivative of \( f \) evaluated at \( x = a \)) by the equation

\[ c_n = \frac{f^{(n)}(a)}{n!} \]

For example, see Problem 6b on Practice Midterm 2C.