Failure to follow the instructions below is a breach of the Stanford Honor Code:

- You may not use or consult any book or notes during the exam.*
- You may not use a calculator or the calculator function on any electronic device during the exam.*
- You may not access any internet-capable electronic device during the exam,* including smartphones and smartwatches, for any reason. These devices must be switched to “airplane mode” and disconnected from all wireless networks (both cellular and wifi) during the exam*.
- You must sit in your assigned seat.
- You may not communicate with anyone other than the course staff during the exam,* or look at anyone else’s solutions.

*“During the exam” is defined as: After you start the exam, and before you turn in the exam and leave the testing site.

- You have 90 minutes to complete this exam. If the course staff must ask you to stop writing or to turn in your exam more than once after time is called, you may receive a score of zero.
- After you have turned in your exam, you may not discuss the contents of this exam with ANYONE other than the course staff until 9:00 PM tonight.
- If you need to make a phone call during the exam, ask a proctor.

I understand and accept these instructions. All smart devices on my person are in airplane mode and disconnected from all wireless networks.

Signature: __________________________

Remember to show your work and justify your answer if required (additional tips are on the next page). Present all solutions in as organized a manner as possible. GOOD LUCK!
Here are some tips:

- If you have time, it's always a good idea to check your work when possible.

- If you get the wrong answer but show your work, you have a better chance of receiving partial credit.

- DO NOT attempt to estimate any of your answers as decimals. For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is exact.

- The very last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem's page.

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**Error bounds from the Alternating Series Test**

If $a_0 - a_1 + a_2 - a_3 + \cdots$ is an alternating series (with all $a_n \geq 0$), that converges by the A.S.T., then the error in the $n$th partial sum is $\leq$ the absolute value of the $(n+1)$th term. That is,

$$|S_\infty - S_n| \leq |a_{n+1}|$$

where $S_\infty = a_0 - a_1 + a_2 - a_3 + \cdots$ and $S_n = a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$.

*You may use this space for extra work.*
Unless otherwise specified, in problems 1–7 you do not need to justify your answer.

1. Suppose that $p$ is a real number. Consider the infinite series \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots. \]

In the blanks below, describe all values of $p$ for which this series converges absolutely, converges conditionally, or diverges.

<table>
<thead>
<tr>
<th>(a) CONVERGES ABS.</th>
<th>(b) CONVERGES COND.</th>
<th>(c) DIVERGES</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p &gt; 1$</td>
<td>$0 &lt; p \leq 1$</td>
<td>$p \leq 0$</td>
</tr>
</tbody>
</table>

2. Simplify the following expression (where $n$ is any integer $\geq 1$):

\[ 1 + 5 \cdot \sum_{k=0}^{n-1} 4^k = 1 + 5(1 + 4 + 4^2 + \cdots + 4^{n-1}) \]

Your final answer should be expressed as a simple formula in terms of $n$.

Your final answer may not include $\sum$ or $\cdots$. Draw a box around your final answer.

\[ 1 + 5 \cdot \sum_{k=0}^{n-1} 4^k = 1 + 5 \cdot \frac{4^n - 1}{4 - 1} \quad \text{(Geom } \sum \text{ formula)} \]

\[ = 1 + \frac{5}{3}(4^n - 1) \]
3. In questions a–e below, write down a \textit{sequence} (a simple formula in terms of $n$ or $k$, for example) or a \textit{series} (in $\sum$ form) that satisfies the given conditions, or state that such a sequence/series does not exist. You do not need to justify your answers.

a. A sequence that is \textit{bounded} but \textit{not} \textit{convergent}.

\[ (-1)^n \]

b. A sequence that is \textit{convergent} but \textit{not} \textit{monotone}.

\[ \frac{(-1)^n}{n} \]

c. A sequence that is \textit{monotone and bounded} but \textit{not} \textit{convergent}.

d. A series that \textit{converges absolutely} but for which the Ratio Test is \textit{inconclusive}.

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \]

e. A series that \textit{diverges} but for which the Divergence Test is \textit{inconclusive}.

\[ \sum_{n=1}^{8} \frac{1}{n} \]
4. The series \[ \sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2 + k}} \] satisfies \[ \sum_{k=1}^{n} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2 + k}} = 1 - \frac{1}{\sqrt{n+1}}. \]

Does the series above converge or diverge? Justify your answer.

It converges!

\[ \sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2 + k}} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2 + k}} = \lim_{n \to \infty} \left[ 1 - \frac{1}{\sqrt{n+1}} \right] = 1 \]

1. Definition of infinite series
   - \( \lim(\text{partial sums}) \)
2. Formula given for \( \sum \).

5. a. Why can’t direct comparison alone be used to determine whether the series \( \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \) converges or diverges? Just give a brief reason—further explanation is not necessary.

Direct comparison requires that

the series have terms that are [eventually] always positive, but \( \sin n \) is negative.

b. Explain how to show that the series \( \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \) converges.

The series \( \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges by direct comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

So \( \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \) converges by Absolute Convergence Test.
6. a. Let \( \{a_k\}_{k=0}^{\infty} \) be a sequence of positive numbers and consider the alternating series

\[
\sum_{k=0}^{\infty} (-1)^k a_k = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + \cdots
\]

When can we conclude that the series above converges by the Alternating Series Test? That is, what properties would the sequence \( \{a_k\} \) need to have?

The sequence \( \{a_k\} \) must be

- decreasing, and
- \( \lim_{k \to \infty} a_k = 0 \)

b. The infinite series below converges by the Alternating Series Test: \( \frac{1}{100000} \) in abs. val.

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{(7n+1)^3} = \left[ 1 - \frac{1}{512} + \frac{1}{3375} - \frac{1}{10648} + \frac{1}{24389} - \frac{1}{46656} + \frac{1}{79507} \right] - \frac{1}{125000} + \frac{1}{185193} - \cdots
\]

Draw a close-bracket [ ] in the right-hand expression above so that the expression between the brackets [ ] is the partial sum of the series with the least number of terms that is guaranteed (by the error bounds on page 2) to estimate the series’ value to within 10^{-5}.

7. To what value does the infinite series \( \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^n}{3 \cdot n!} \) converge?

No justification necessary. Draw a box around your final answer.

\[
= \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} = \frac{1}{3} e^{-2} = 1/(3e^2)
\]
8. Find the interval of convergence of the following power series: \( \sum_{n=0}^{\infty} \frac{(x-2)^n}{5^n(2n+1)} \).

Show your work and draw a box around your final answer.

(1) Center is at 2.

(2) Ratio Test:

\[
L(x) = \lim_{n \to \infty} \left| \frac{(x-2)^{n+1}}{5^{n+1}(2n+3)} \cdot \frac{5^n(2n+1)}{(x-2)^n} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{x-2}{5} \cdot \frac{2n+1}{2n+3} \right|
\]

\[
= \frac{|x-2|}{5} \cdot \lim_{n \to \infty} \left| \frac{2n+1}{2n+3} \right| = 1
\]

\[
\frac{|x-2|}{5} = 1 \rightarrow |x-2| = 5
\]

\[
\rightarrow x-2 = 5 \quad \text{or} \quad x-2 = -5
\]

\[
\rightarrow x = 7 \quad \text{or} \quad x = -3
\]

(3) Test endpoints:

At \( x = 7 \):

\[
\sum_{n=0}^{\infty} \frac{(7-2)^n}{5^n(2n+1)} = \sum_{n=0}^{\infty} \frac{5^n}{5^n(2n+1)} = \sum_{n=0}^{\infty} \frac{1}{2n+1}
\]

\[
= \frac{\pi}{2}, \quad \text{diverges!}
\]

At \( x = -3 \):

\[
\sum_{n=0}^{\infty} \frac{(-3-2)^n}{5^n(2n+1)} = \sum_{n=0}^{\infty} \frac{(-5)^n}{5^n(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}
\]

Final answer:

\([-3, 7)\]

\[\text{or} \quad -3 < x \leq 7\]
For the six series 9–14 below, fill in the required information:

(a) Whether the series converges or diverges;
(b) Which test(s) you ultimately used to determine convergence/divergence; and
(c) Additional info: If you used the *integral test*, write down the value of the relevant integral (or that it diverges); if you used a *direct or limit comparison test*, state what series you compared the given series to; if you used the *ratio test*, state the value of $L$ you found. (Leave blank for other tests.)

The next page can be used for any extra work, but it will not be graded. You do *not* need to specify absolute vs. conditional convergence.

### 9. \[ \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \]

<table>
<thead>
<tr>
<th>(a) Converge/diverge?</th>
<th>(b) Test(s) used</th>
<th>(c) Additional info</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converge</td>
<td>Ratio</td>
<td>[ L = \frac{1}{4} ]</td>
</tr>
</tbody>
</table>

### 10. \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \]

<table>
<thead>
<tr>
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<th>(b) Test(s) used</th>
<th>(c) Additional info</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converge</td>
<td>Alt. Series</td>
<td></td>
</tr>
</tbody>
</table>

### 11. \[ \sum_{n=0}^{\infty} \frac{\cos(2\pi n)}{\sqrt{4n^2 + 1}} \]

\[ \sum_{n=0}^{\infty} \frac{1}{\sqrt{4n^2 + 1}} \quad \text{b/c} \quad \cos(2\pi n) = 1 \text{ always} \]

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</tr>
</thead>
<tbody>
<tr>
<td>Diverge</td>
<td>Lim. Comp.</td>
<td>[ \sum_{n=1}^{8} \frac{1}{n} ]</td>
</tr>
</tbody>
</table>

### 12. \[ \sum_{n=2}^{\infty} \frac{\ln n}{n^{1.25}} \]

\[ \frac{\ln n}{n^{1.25}} < \frac{n^{0.01}}{n^{1.25}} = \frac{1}{n^{1.24}} \]

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</tr>
</thead>
<tbody>
<tr>
<td>Converge</td>
<td>Lim. Comp.</td>
<td>[ \sum_{n=1}^{\infty} \frac{1}{n^{1.24}} ]</td>
</tr>
</tbody>
</table>

*(not the only choice!)*
13. \( \sum_{n=0}^{\infty} \frac{4n+3}{\sqrt{5n^3} + n + 1} \) terms \( \propto \frac{1}{\sqrt{n}} \)

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<tbody>
<tr>
<td>Diverge</td>
<td>Lim. Comp.</td>
<td>( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} )</td>
</tr>
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</table>

14. \( \sum_{n=0}^{\infty} \frac{n^{100}}{100^n} \)

<table>
<thead>
<tr>
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<th>(b) Test(s) used</th>
<th>(c) Additional info</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converge</td>
<td>Ratio</td>
<td>( L = \frac{1}{100} )</td>
</tr>
</tbody>
</table>

The remaining space is provided for any extra work. If you think this work is important to one of your solutions, please indicate that on the page of the relevant problem (otherwise we won’t know to look!).
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If you need additional space for work beyond this, ask a proctor.