Math 21, Fall 2017 — Schaeffer/Kemeny  
Final Exam (December 11th, 2017)

<table>
<thead>
<tr>
<th>Last/Family Name</th>
<th>First/Given Name</th>
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Failure to follow the instructions below will constitute a breach of the Stanford Honor Code:

- **You may not write anything on this page other than your name, seat #, and signature.**
- You may not use a calculator or any notes or book during the exam.*
- You may not access your phone or any other electronics during the exam for any reason. *
- You must sit in your assigned seat during the exam.*
- You may not communicate with anyone other than the course staff during the exam, or look at anyone else’s solutions during the exam.*
- **Additionally, you may not discuss the contents of this exam with ANYONE other than the course staff until 10:00 PM tonight.**
- You have 180 minutes to complete this exam. If the course staff must ask you to stop writing or to turn in your exam more than once after time is called, you may receive a score of zero.

*During the exam is defined as “before you have handed in your exam and left the testing site.”

I understand and accept these instructions.

Signature: _______________________________________________________

Remember to show your work and justify your answer if required. Present all solutions in as organized a manner as possible.

The last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page. **Do not detach the last page of the exam.**

GOOD LUCK!
Here are some tips:

- If you have time, it’s always a good idea to check your work when possible.
- If you get the wrong answer but show your work, you have a better chance of receiving partial credit.
- **DO NOT** attempt to estimate any of your answers as decimals. For example, \( 1 - \frac{1}{\pi} \) is a much better answer than 0.682, because it is exact.
- The last page of the exam are blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

**Integral table entries you may need**

\[
\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C \quad \int \frac{du}{(u - r)(u - s)} = \frac{1}{r - s} \ln \left| \frac{u - r}{u - s} \right| + C
\]

\[
\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C
\]

\[
\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \left( \frac{u}{a} \right) + C
\]

In the entries above, \( a, r, s \) are constants such that \( a \neq 0 \) and \( r \neq s \).

**Values of arcsine and arctangent**

The table below gives important values of the arcsine and arctangent functions:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \frac{1}{2} )</th>
<th>( \frac{1}{\sqrt{3}} )</th>
<th>( \frac{\sqrt{3}}{2} )</th>
<th>1</th>
<th>( \sqrt{3} )</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \arcsin x )</td>
<td>0</td>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{\pi}{3} )</td>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{\pi}{4} )</td>
<td>undef.</td>
</tr>
<tr>
<td>( \arctan x )</td>
<td>0</td>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{\pi}{6} )</td>
<td>( \pi/3 )</td>
<td>( \pi/4 )</td>
<td>( \pi/4 )</td>
<td>( \pi/3 )</td>
</tr>
</tbody>
</table>

For negative values: \( \arcsin(-x) = -\arcsin(x) \) and \( \arctan(-x) = -\arctan(x) \).

Blank entries in the table are not “nice” multiples of \( \pi \).

**Bounds from the integral and alternating series tests**

- If \( \sum_{n=a}^{\infty} f(n) \) converges by the integral test, then
  \[
  \int_{a}^{\infty} f(x) \, dx \leq \sum_{n=a}^{\infty} f(n) \leq f(a) + \int_{a}^{\infty} f(x) \, dx
  \]
- If \( \sum_{n=0}^{\infty} (-1)^n a_n \) converges by the alternating series test, then
  \[
  |\text{error in the Nth partial sum}| \leq |\text{the (N + 1)th term in the series}|
  \]
Section 1 — Improper Integrals, Sequences, Geometric Series

1.1 Fill in the blanks below with the appropriate asymptotic relation: $\prec$, $\asymp$, or $\succ$ (as $x$ or $n \to \infty$).

a. $e^{x^2} \quad \prec \quad (e^x)^2$

b. $3^{-x} \quad \prec \quad (1/2)^x$

c. $n! \quad \prec \quad 1000000^n$

d. $\frac{x}{\sqrt{x^2 + 1}} \quad \prec \quad \frac{1}{x^2 + 1}$

e. $x^2 \cdot e^x \quad \prec \quad 3^x$

f. $x^2 + e^x \quad \prec \quad \sqrt{x} + e^x$

g. $\frac{1}{x^2 + e^x} \quad \prec \quad \frac{1}{\sqrt{x} + e^x}$

h. $\frac{1}{x^2} + \frac{1}{e^x} \quad \prec \quad \frac{1}{\sqrt{x}} + \frac{1}{e^x}$

i. $x^{0.000001} \quad \prec \quad \ln x$

j. $n! \quad \prec \quad n^n$

1.2 a. For which values of $p$ does \( \int_1^\infty \frac{1}{x^p} \, dx \) converge? Describe all such values.

b. For which values of $p$ does \( \int_0^1 \frac{1}{x^p} \, dx \) converge? Describe all such values.
1.3 Partario is attempting to evaluate the integral \[ \int_{-1}^{0} \frac{e^{1/x}}{x^2} \, dx \], illustrated below:

By substituting \( u = \frac{1}{x} \), he correctly finds \[ \int \frac{e^{1/x}}{x^2} \, dx = -e^{1/x} + C. \]

Because the integrand is not defined at \( x = 0 \), Partario continues as follows:

\[
\int_{-1}^{0} \frac{e^{1/x}}{x^2} \, dx = \lim_{b \to 0} \int_{-1}^{b} \frac{e^{1/x}}{x^2} \, dx = \lim_{b \to 0} \left[ -e^{1/x} \right]_{-1}^{b} \\
= e^{-1} - \lim_{b \to 0} \left[ e^{1/b} \right] = e^{-1} - e^{\infty} = -\infty
\]

So the integral diverges!

Is Partario correct or incorrect? If you believe he is correct, write PARTARIO IS CORRECT. Otherwise, explain what went wrong in Partario’s solution and write down the correct answer.
In problems 1.4–1.6, evaluate the improper integral, showing all work. If you believe the integral diverges, write DIVERGES. Some integration formulas and values of inverse trigonometric functions are available for your use on pg 2.

Draw boxes around your final answers.

1.4 \( \int_{-\infty}^{\infty} \frac{dx}{4x^2 + 1} \)

1.5 \( \int_{-4}^{4} \frac{1}{x^3} \, dx \)
1.6 \( \int_{1}^{\infty} x^2 e^{-x^3} \, dx \)

1.7 Does the integral \( \int_{0}^{1} \ln x \, dx \) converge or diverge? You do not need to justify your answer.

*Draw a box around your final answer.*
1.8 Does the integral $\int_0^\infty \frac{\ln(x + 1)}{x^{7/6}} \, dx$ converge or diverge? Carefully justify your answer.

Draw a box around your final answer.

Hint: If $\frac{1}{2} \leq p \leq 1$, then $0 \leq \ln(x + 1) \leq x^p$ for all real numbers $x \geq 0$. 
1.9  a. Evaluate the sum $6^3 + 6^4 + 6^5 + \cdots + 6^{100}$. You may leave your answer unsimplified, but it may not contain more than one $+$ or $-$ sign, and it may not contain $\sum$.

    Draw a box around your final answer.

b. Evaluate the geometric series $\frac{1}{4} - \frac{1}{4^2} + \frac{1}{4^3} - \frac{1}{4^4} + \cdots$. (Note: The series is alternating.)

c. Express the repeating decimal $0.021021021\overline{021}$ as a fraction of whole numbers in lowest terms.

    Draw a box around your final answer.
Section 2 — Sequences, Series, Power Series, and Taylor Series

2.1  a,b. Write down an example of a sequence that is bounded but does not converge, or state that no such sequence exists.

c. Write down an example of a sequence that is bounded and monotone but does not converge, or state that no such sequence exists.
2.2 Suppose that the series \( \sum_{n=0}^{\infty} a_n \) converges conditionally (and not absolutely). Which of the following statements must be true? Circle all true statements.

i. If the limit \( L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \) exists, then \( L = 1 \).

ii. The series \( \sum a_n \) is alternating: Consecutive nonzero terms always have opposite signs.

iii. The sequence \( \{a_n\} \) contains both infinitely many positive terms as well as infinitely many negative terms.

iv. The limit of the sequence \( \{a_n\} \) is 0.

v. The limit of the sequence \( \{|a_n|\} \) is 0.

vi. The limit \( \lim_{n \to \infty} \frac{1}{n^2} \), provided it exists and is finite, is equal to zero.

vii. The limit \( \lim_{N \to \infty} \sum_{n=0}^{N} a_n \) exists and is finite.

viii. None of the above.

2.3 Determine whether each series (a–e) below converges or diverges. If the series converges (either absolutely or conditionally), circle it. If the series diverges, draw an X through it.

Below each series, fill in the blank with the roman numeral I–VII corresponding to the reason you chose convergence/divergence.

a. \( \sum_{n=1}^{\infty} \frac{\sin n}{n^4} \)  

b. \( \sum_{n=1}^{\infty} \frac{\sin(n\pi)}{n} \)

c. \( \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} \)

d. \( \sum_{n=1}^{\infty} \frac{1 + (\sin n)^2}{n} \)

e. \( \sum_{n=1}^{\infty} \frac{1}{1 + (\sin n)^2} \)

Under each of the five series, make sure to include one of the following reasons:

I. The series converges by comparison with the harmonic series.

II. The series diverges by comparison with the harmonic series.

III. The series diverges by the divergence test (terms do not tend to zero).

IV. The series converges by the alternating series test.

V. \( \sum |a_n| \) converges, so \( \sum a_n \) converges too (absolute convergence guarantees convergence).

VI. The terms of the series are all zero, so the series converges.

VII. The series converges by the integral test.

VIII. None of the above.
2.4 Below are ten infinite series (a–j).

- If the series converges (either absolutely or conditionally), circle it.
- If the series diverges, draw an X through it.

\[
\begin{align*}
\text{a. } & \sum_{n=0}^{\infty} \frac{1}{3n + 1} \\
\text{b. } & \sum_{n=0}^{\infty} |\sin(n)| \\
\text{c. } & \sum_{n=0}^{\infty} \frac{n^2}{\sqrt{4n^9 + 1}} \\
\text{d. } & \sum_{n=2}^{\infty} \frac{1}{n \ln n} \\
\text{e. } & \sum_{n=0}^{\infty} \frac{(-1)^n}{n^{0.001}} \\
\text{f. } & \sum_{n=0}^{\infty} \frac{n!^3}{(3n)!} \\
\text{g. } & \sum_{n=0}^{\infty} \frac{2^n \cdot n^2}{3^n + 1} \\
\text{h. } & \sum_{n=1}^{\infty} \frac{1 + 2 \cdot (-1)^n}{n^3} \\
\text{i. } & \sum_{n=1}^{\infty} \frac{\ln n}{n^4} \\
\text{j. } & \sum_{n=0}^{\infty} \frac{n!}{100^n}
\end{align*}
\]

You do not need to show your work.

|——— SPACE BELOW FOR SCRATCH WORK (MORE AT END OF EXAM) ————|
2.5 Let $F(x)$ be the power series $\sum_{n=0}^{\infty} \frac{(-1)^n (x + 1)^{3n+1}}{8^n \cdot (3n + 1)}$. What is $F(x)$’s interval of convergence?

Show all work and draw a box around your final answer.
2.6 In (a–d) write down the Taylor series for the given function at $x = 0$ in $\sum$ form.

a. $\cos x$

b. $\ln(1 + x)$

c. $x - \arctan x$

d. $\frac{x}{1 + x^3}$
2.7 Below are four equations involving infinite series, each with one unknown quantity:

\[
\sum_{n=0}^{\infty} \frac{X^n}{n!} = 3 \quad \sum_{n=0}^{\infty} \frac{(-1)^n Y^{2n}}{(2n)!} = 2 \quad \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n + 1)} = Z \quad \sum_{n=0}^{\infty} W^n = -1
\]

In the spaces below, write down values for \(X\), \(Y\), \(Z\), and \(W\) that make each of the equations above true, or write DNE if no such value exists.

*Your answers may not contain any unresolved inverse trig functions. A table of inverse trig values is available for your use on pg 2.*

a. \(X = \quad \)__________

b. \(Y = \quad \)__________

c. \(Z = \quad \)__________

d. \(W = \quad \)__________
Section 3 — Applications of Taylor Series

3.1 Quintana is studying a mystery function \( G(t) \), which she knows to be smooth (infinitely differentiable) at \( t = 0 \). She is attempting to approximate the value of \( G(0.4) \) using the quadratic approximation (2nd degree Taylor polynomial), \( P_2(x) \) at \( x = 0 \). To make sure her estimate is not too far off from the actual value, she uses the Lagrange error bounds, which take the form

\[
|G(0.4) - P_2(0.4)| \leq \frac{M}{2} \cdot ?
\]

a. Help Quintana out by complete the Lagrange error bounds she would use in this case, (that is, rewrite the above, filling in the ?s). You may leave \( M \) as is.

b. Explain what \( M \) is in the Lagrange error bounds, as written above.
3.2 The Taylor series for the tangent function at \( x = 0 \) has a rather complicated closed form, which is why we never had you learn it (unlike those for sine and cosine). The first few terms of the series are

\[
\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots
\]

and the series converges on \((-\frac{\pi}{2}, \frac{\pi}{2})\).

Using this information, or whatever method you prefer, answer the questions below.

a. Using the 3rd degree Taylor polynomial of \( \tan x \) at \( x = 0 \), write down an estimate for \( \int_{0}^{1} \tan x \, dx \).

You do not need to simplify your answer, which should be a sum of fractions.

Draw a box around your final answer.

b. If you instead use the 4th degree Taylor polynomial of \( \tan x \) at \( x = 0 \) to estimate the integral, will your estimate be better, worse, or the same?
c. Let \( f(x) = \tan x - \sin x \). What is the value of \( f^{(5)}(0) \)?

\[ \text{Draw a box around your final answer.} \]

d. Evaluate the limit \( \lim_{x \to 0} \left[ \frac{\tan x - \sin x}{x^3} \right] \). \( \text{Draw a box around your final answer.} \)

3.3 Consider the definite (proper) integral \( \int_0^1 \sin(x^2) \, dx \).

a. Express the value of the integral above as an \textit{infinite series} (using \( \sum \)).
b. The series you obtained in the previous part should be alternating (if it’s not, you should check your work!). Using the bounds from the alternating series test (on pg 2), write down a finite sum of fractions that is guaranteed to estimate the value of the integral to within $10^{-3} = \frac{1}{1000}$.

Your answer may not contain $\sum$ or $\cdots$.

Draw a box around your final answer.

3.4 Suppose $y = f(x)$ is a solution to the differential equation

$$y'' + 4xy' + y = 0$$

satisfying $f(0) = 1$ and $f'(0) = -1$. Assume that $f(x)$ is smooth (infinitely differentiable) at $x = 0$, so we can find its Taylor series there.

a. Writing $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ for the Taylor series of $f(x)$ at $x = 0$, what is the Taylor series for $f'(x)$ at $x = 0$? Express your answer as an infinite (power) series.
b. What is the Taylor series for $f''(x)$ at $x = 0$? You may express your answer either as an infinite (power) series, or as (the first three nonzero terms) $\cdots$.

c. What are $c_0$ and $c_1$ equal to?

d. Find $P_4(x)$, the 4th degree Taylor polynomial for $f(x)$ at $x = 0$.

*Draw a box around your final answer.*