Failure to follow the instructions below will constitute a breach of the Honor Code:

- You may not use consult any book or notes during the exam.*
- You may not use a calculator or the calculator function on any electronic device during the exam.*
- You may not access any internet-enabled electronic device during the exam*.
- *“During the exam” is defined as: After you start the exam, and before you turn in the exam and leave the testing site.
- You may not detach any page of this exam.
- You may not write anything on this page of the exam (except to complete the identifying information above and your signature).
- If you need to check the time, you may ask your proctor.
- If you have a question on the exam material, your proctor has the instructors’ phone numbers. They will contact us and you may speak with us over the phone.
- You have 3 hours to complete this exam.
- You may not discuss or share the contents, material, or your emotional reaction to this exam (e.g., whether you found it hard or easy) with ANYONE other than the course staff until 10:00 PM Pacific tonight.

I understand and accept these instructions.

Signature: _______________________________________________________

Remember to show your work and justify your answer if required. Present all solutions in as organized a manner as possible. GOOD LUCK!
Here are some tips:

- If you have time, it’s always a good idea to _check your work_ when possible.
- If you get the wrong answer but _show your work_, you have a better chance of receiving partial credit.
- _DO NOT_ attempt to estimate any of your answers as decimals. For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is _exact_.
- The last page of the exam are blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

**Integral table entries you may need**

$$
\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C,
\int \frac{du}{(u - r)(u - s)} = \frac{1}{r - s} \ln \left| \frac{u - r}{u - s} \right| + C
$$

$$
\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C
$$

$$
\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \left( \frac{u}{a} \right) + C
$$

In the entries above, $a, r, s$ are constants such that $a \neq 0$ and $r \neq s$.

**Values of arcsine and arctangent**

The table below gives important values of the arcsine and arctangent functions:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{1}{\sqrt{3}}$</th>
<th>$\frac{1}{\sqrt{2}}$</th>
<th>$\sqrt{3}$</th>
<th>1</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>arcsin $x$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/6$</td>
<td>$\pi/4$</td>
<td>$\pi/3$</td>
<td>$\pi/2$</td>
<td>undef.</td>
</tr>
<tr>
<td>arctan $x$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/6$</td>
<td>$\pi/4$</td>
<td>$\pi/3$</td>
<td>$\pi/2$</td>
<td>$\pi/2$</td>
</tr>
</tbody>
</table>

For negative values: $\arcsin(-x) = -\arcsin(x)$ and $\arctan(-x) = -\arctan(x)$.
Blank entries in the table are not “nice” multiples of $\pi$.

**Error bounds from the alternating series tests**

If $\sum_{n=0}^{\infty} (-1)^n a_n$ converges by the alternating series test, then

$$
|\text{error in the } M\text{th partial sum of the series}| \leq |\text{the } (M + 1)\text{th term in the series}|
$$
Section 1: Improper integrals, geometric sums and series

1. a. For which values of \( p \) does the integral \( \int_{2}^{\infty} \frac{1}{x^p} \, dx \) converge? Describe all such values.

b. For which values of \( p \) does the integral \( \int_{0}^{1} \frac{1}{x^p} \, dx \) converge? Describe all such values.

c. For which values of \( a \) does the integral \( \int_{-\infty}^{\infty} e^{ax} \, dx \) converge? Describe all such values.
2. In a.–i. below, fill in the blank with the correct asymptotic relation \( \ll, \gg, \) or \( \sim \).

a. \( n! + 2^n \ll n^n \)

b. \( x^5 + 5^x \ll (x^3 + 2^x)^2 \)

c. \( e^n \gg 3^n \)

d. \( \ln x \ll x^{1/3} \)

e. \( \frac{\sqrt{x^4 + x + 17}}{(2x)^6 + 35)^{1/3}} \ll 55 \)

f. \( x^{-2} \ll 2^{-x} \)

g. \( \ln(x^{1/12}) \ll \ln(x^{-1}) \)

h. \( x \cdot \frac{1}{\ln(\ln(\ln(x)))} \ll 1000000^{100000} \cdot x \)

i. \( \arctan(x) \ll \frac{1}{1/x + 1} \)

3. Circle **all** true statements:

i. \( \int_{-1}^{1} \frac{1}{x} \, dx = 0. \)

ii. \( \int_{-1}^{1} x^3 \, dx = 0. \)

iii. \( \int_{-1}^{1} \frac{1}{x^{1/3}} \, dx = 0. \)

iv. None of the above are true.
4. Evaluate: \[ \int_0^\infty \left( \frac{2x}{1 + x^2} - \frac{1}{\frac{1}{2}x + 1} \right) \, dx. \]

Show all work and box your final answer. If you believe the integral diverges, say so.

5. Evaluate: \[ \int_3^\infty \frac{1}{x^2 + 9} \, dx. \]

Show all work and box your final answer. If you believe the integral diverges, say so.
6. Consider the improper integral
\[ \int_{1/3}^{\infty} \frac{x}{\ln x \cdot \sqrt[3]{(x-3)(x-5)(x-7)}} \, dx \]

a. Express this improper integral as a sum of simple improper integrals.

Remember that a “simple” improper integral is an improper integral that has only a single issue and that issue is located at one of the endpoints.

To simplify notation, you may use the abbreviation \( \int_a^b \) for \( \int_a^b \frac{x}{\ln x \cdot \sqrt[3]{(x-3)(x-5)(x-7)}} \, dx. \)

b. Does this improper integral converge or diverge?
7. Does the integral \[ \int_0^\infty \frac{1 + \cos 3x}{e^x} \, dx \] converge or diverge? \textit{Justify your answer.}

8. Does the integral \[ \int_0^1 \frac{1}{x^{3/2} \ln(1 + x) + \sqrt{x}} \, dx \] converge or diverge? \textit{Justify your answer.}
9. Modern *humans* typically use base-10 *decimal* expansions to write numbers. That means that each digit corresponds to a multiple of a power of 10: for example,

\[132.37 = 100 + 30 + 2 + \frac{3}{10} + \frac{7}{100} = 1 \cdot 10^2 + 3 \cdot 10^1 + 2 \cdot 10^0 + 3 \cdot 10^{-1} + 7 \cdot 10^{-2}.\]

*Computers*, on the other hand, typically use base-2 *binary* expansions. In binary, instead of using powers of 10, each digit corresponds to a power of 2, and the only digits are 0 and 1. So, for example, the decimal number 5.5 is written in binary as 101.1 because

5.5 in decimal = \(5 + \frac{5}{10} = 4 + 0 + 1 + \frac{1}{2} = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 + 1 \cdot 2^{-1} = 101.1\) in binary.

a. The binary number .101010101 is equal to

\[
\frac{1}{2^1} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{0}{2^4} + \frac{1}{2^5} + \frac{0}{2^6} + \frac{1}{2^7} + \frac{0}{2^8} + \frac{1}{2^9}
\]

Express the number above as a fraction in lowest terms. *Hint:* \(2^{10} = 1024\) and \(1023 = 3 \cdot 341\).

b. Write the binary number 111.\(\overline{1}\) = 111.111\ldots\) (the 1s repeat forever) as a whole number or as a fraction in lowest terms.
Section 2: Sequences and series, convergence tests, power series

10. Suppose that $s$ is a real number. Consider the infinite series

$$
\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots.
$$

In the blanks below, describe all values of $s$ for which this series converges absolutely, converges conditionally, or diverges.

<table>
<thead>
<tr>
<th>CONVERGES ABS.</th>
<th>CONVERGES COND.</th>
<th>DIVERGES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

11. Let \( \{a_k\} = \{a_1, a_2, a_3, \ldots\} \) be a sequence, and let \( \{S_n\} \) be the associated sequence of partial sums:

$$
S_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n
$$

(a) Circle the true statement(s) below.

i. If \( \{a_k\} \) converges, then \( \{|a_k|\} \) must also converge.

ii. If \( \{|a_k|\} \) converges, then \( \{a_k\} \) must also converge.

iii. If \( \{a_k\} \) converges, then \( \{S_n\} \) must also converge.

iv. If \( \{S_n\} \) converges, then \( \{a_k\} \) must also converge.

v. If \( \{a_k\} \) is a sequence of positive numbers and \( \{S_n\} \) is bounded, then \( \{S_n\} \) must converge.

vi. None of the above.
In this course we discussed in detail how to compute the error in partial sums of *alternating series*. But what about convergent series whose terms are all positive? One way to estimate the error in this case is to use the bounds from the *integral test*.

Suppose that \( f \) is positive, decreasing, and continuous on \([1, \infty)\), that \( \int_1^\infty f(x) \, dx \) converges. Then \( \sum f(k) \) converges and we have for every integer \( n \geq 1 \) the following inequality:

\[
\int_n^\infty f(x) \, dx \leq \sum_{k=n}^{\infty} f(k) \leq f(n) + \int_n^{\infty} f(x) \, dx.
\]

a. Briefly explain how to justify the inequality on the left above (drawing a picture might help!).

b. Let \( S = \sum_{k=1}^{n-1} f(k) \). Briefly explain how we can use the inequalities above to conclude that

\[
\underbrace{S + \int_n^\infty f(x) \, dx}_{A_n} \leq \underbrace{\sum_{k=1}^{\infty} f(k)}_{\text{error}} \leq S + f(n) + \underbrace{\int_n^{\infty} f(x) \, dx}_{B_n}.
\]
c. With $A_n$ and $B_n$ as defined in (b), briefly explain why we know that $B_n - A_n \to 0$ as $n \to \infty$.

Note: This means that the range of values $[A_n, B_n]$ where $\sum_{k=1}^\infty f(k)$ lies gets smaller and smaller as $n \to \infty$. This allows us to estimate $\sum_{k=1}^\infty f(k)$ to any desired precision.

13. Let $F(x) = \sum_{n=0}^\infty a_n x^n$ and $G(x) = \sum_{n=0}^\infty b_n x^n$ with all $a_n > 0$ and all $b_n > 0$. Suppose that

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2} \quad \text{and} \quad \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = 1$$

Let $x_0$ be a real number. Circle all of the following statements that are true:

i. If $x_0 < 0$, then the series $F(x_0)$ is alternating.

ii. If $F(x)$ converges at $x = x_0$, then $G(x)$ converges at $x = x_0$ too.

iii. If $G(x)$ converges at $x = x_0$, then $F(x)$ converges at $x = x_0$ too.

iv. The power series $F(x) + G(x) = \sum_{n=0}^\infty (a_n + b_n)x^n$ converges at $x = 1/3$.

v. The series $F(-1)$ converges absolutely.

vi. None of the above.
14. Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges conditionally.

a. Explain briefly what this means.

b. Again: Suppose that $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Using this information, decide in (i–vii) whether each of the given statements is ALWAYS TRUE, SOMETIMES (but not always) TRUE, or ALWAYS FALSE and circle your answers.

i. The sequence $\{|a_n|\}$ converges.

**ALWAYS TRUE**  **SOMETIMES TRUE**  **ALWAYS FALSE**

ii. The series $\sum_{n=1}^{\infty} (a_n + 1)$ converges.

**ALWAYS TRUE**  **SOMETIMES TRUE**  **ALWAYS FALSE**

iii. The series $\sum_{n=1}^{\infty} a_n$ is an alternating series.

**ALWAYS TRUE**  **SOMETIMES TRUE**  **ALWAYS FALSE**

iv. The sequence $\{a_n\}$ has infinitely many negative terms.

**ALWAYS TRUE**  **SOMETIMES TRUE**  **ALWAYS FALSE**

v. The sequence $\{a_n\}$ satisfies $|a_n| \geq \frac{1}{n}$.

**ALWAYS TRUE**  **SOMETIMES TRUE**  **ALWAYS FALSE**

vi. The sequence $\{a_n\}$ satisfies $|a_n| \geq \frac{1}{n^2}$.

**ALWAYS TRUE**  **SOMETIMES TRUE**  **ALWAYS FALSE**
15. For the series (a–h) below, determine whether the series CONVERGES or DIVERGES and circle your answer. If an extra space is provided for $L$, fill it in with the value you would find if you applied the ratio test (regardless of whether that is a “good” test to apply for that series).

You do not need to show your work or specify absolute vs. conditional convergence.

The next page and a half have been left blank for extra work. It will not be graded.

\[
\begin{align*}
\text{a. } & \sum_{n=2}^{\infty} \frac{1}{n^2 (\ln n)^2} & \text{b. } & \sum_{n=1}^{\infty} \frac{\sqrt{n^4 + 4^n}}{n^4 + n + 1} \\
& \text{CONVERGES} & \text{CONVERGES} & \text{DIVERGES} \text{ DIVERGES}
\end{align*}
\]

\[
\begin{align*}
\text{c. } & \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n + 1)^{1/3}} & \text{d. } & \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \\
& \text{CONVERGES} & \text{CONVERGES} & \text{DIVERGES} \text{ DIVERGES}
\end{align*}
\]

\[
\begin{align*}
L &= \underline{\phantom{0}} & L &= \underline{\phantom{0}}
\end{align*}
\]

\[
\begin{align*}
\text{e. } & \sum_{n=0}^{\infty} (\cos 1)^n & \text{f. } & \sum_{n=0}^{\infty} 2^n \sin(\pi n) \\
& \text{CONVERGES} & \text{CONVERGES} & \text{DIVERGES} \text{ DIVERGES}
\end{align*}
\]

\[
\begin{align*}
L &= \underline{\phantom{0}} & L &= \underline{\phantom{0}}
\end{align*}
\]

\[
\begin{align*}
\text{g. } & \sum_{n=0}^{\infty} \frac{n^{10000}}{1.0001^n} & \text{h. } & \sum_{n=0}^{\infty} \frac{n!}{(n+2)!} \\
& \text{CONVERGES} & \text{CONVERGES} & \text{DIVERGES} \text{ DIVERGES}
\end{align*}
\]

\[
\begin{align*}
L &= \underline{\phantom{0}} & L &= \underline{\phantom{0}}
\end{align*}
\]
Left blank for additional work on problem 6.
16. a. Write down a power series that represents (converges to) either \( \ln x \) or \( \ln(x + 1) \) on its interval of convergence (be sure to specify which of \( \ln x \) or \( \ln(x + 1) \) you chose).

b. What is the interval of convergence of the power series you wrote down above?
Section 3: Taylor series and applications

17. Below, solve for $X$, $Y$, and $Z$, or state that no solutions exist. *It is possible that multiple solutions exist, in which case you only have to specify one solution for full credit.*

a. Solve $\sum_{n=0}^{\infty} \frac{3^n X^n}{n!} = 5$ for $X$ or state that no solutions exist.

b. Solve $\sum_{n=0}^{\infty} Y^n = \frac{1}{2}$ for $Y$ or state that no solutions exist.

c. Solve $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^n (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n Z^{2n+1}}{(2n + 1)!}$ for $Z$ or state that no solutions exist.
18. Evaluate the limit \( \lim_{x \to 0} \frac{(\sin x)(\arctan x)(1 - \sqrt{1 + x})^2}{x^2(\cos x - 1)} \).

You may use whatever method you like, but you must show your work.

Draw a box around your final answer.
19. a. Evaluate the integral \( \int_0^1 \frac{1 - \cos(x^3)}{x} \, dx \) in terms of a convergent infinite series.

Your answer should be presented in \( \sum \) form. You do not need to prove that your series converges.
b. Estimate the value of the integral in (a) so that the error in your estimate is $< 10^{-3}$. 
Your answer should be a finite sum/difference of fractions. It may not include $\sum$ or $\cdots$. 
Draw a box around your final answer.

20. Suppose that the power series $y = F(x) = \sum_{n=0}^{\infty} c_n x^n$ satisfies the differential equation 
$$y'' = 4y + (x + 1)^2 \text{ with } y(0) = -1 \text{ and } y'(0) = 2.$$ 
What is the value of $c_3$, the coefficient of $x^3$, in $F(x)$? Draw a box around your final answer.
Breadth questions!

Circle the correct answers. Four out of five must be answered correctly for full credit.

a. The gravitational interactions of three or more objects always display periodic behavior.

   TRUE          FALSE

b. Which Greek letter is used to denote the limit of $\sum_{k=1}^{n} \frac{1}{k} - \ln n$ as $n \to \infty$?

   $\pi$       $\gamma$       $\Omega$

c. What phenomenon causes ringing artifacts in images and can lead to medical misdiagnoses?

   Gibbs Phenomenon   Ultraviolet Catastrophe   Poisson Distribution

d. To what number does the ratio of consecutive Fibonacci numbers tend?

   $\frac{\pi^2}{6}$   $\frac{1 - \sqrt{5}}{2}$   $\frac{1 + \sqrt{5}}{2}$

e. What is the name of the function $f(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ defined for $s > 1$?

   harmonic series   Riemann zeta   Bessel function

The remaining space is blank for extra work.