Math 21, Fall 2018 — Howe/Schaeffer
Final Exam (December 10th, 2018)

<table>
<thead>
<tr>
<th>Last/Family Name</th>
<th>First/Given Name</th>
<th>Seat #</th>
<th>Exam #</th>
</tr>
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</tbody>
</table>

Failure to follow the instructions below will constitute a breach of the Honor Code:

• You may not use consult any book or notes during the exam.*
• You may not use a calculator or the calculator function on any electronic device during the exam.*
• You may not access any internet-enabled electronic device during the exam*.
• “During the exam” is defined as: After you start the exam, and before you turn in the exam and leave the testing site.
• You may not detach any page of this exam.
• You may not write anything on this page of the exam (except to complete the identifying information above and your signature).
• If you need to check the time, you may ask your proctor.
• If you have a question on the exam material, your proctor has the instructors’ phone numbers. They will contact us and you may speak with us over the phone.
• You have 3 hours to complete this exam.
• You may not discuss or share the contents, material, or your emotional reaction to this exam (e.g., whether you found it hard or easy) with ANYONE other than the course staff until 10:00 PM Pacific tonight.

I understand and accept these instructions.

Signature: _______________________________________________________

Remember to show your work and justify your answer if required. Present all solutions in as organized a manner as possible. GOOD LUCK!
Here are some tips:

• If you have time, it’s always a good idea to check your work when possible.

• If you get the wrong answer but show your work, you have a better chance of receiving partial credit.

• DO NOT attempt to estimate any of your answers as decimals. For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is exact.

• The last page of the exam are blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

Integral table entries you may need

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C$$

$$\int \frac{du}{(u - r)(u - s)} = \frac{1}{r - s} \ln \left| \frac{u - r}{u - s} \right| + C$$

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \left( \frac{u}{a} \right) + C$$

In the entries above, $a, r, s$ are constants such that $a \neq 0$ and $r \neq s$.

Values of arcsine and arctangent

The table below gives important values of the arcsine and arctangent functions:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{1}{\sqrt{3}}$</th>
<th>$\frac{1}{\sqrt{2}}$</th>
<th>$\frac{\sqrt{3}}{2}$</th>
<th>1</th>
<th>$\sqrt{3}$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\arcsin x$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/3$</td>
<td>$\pi/2$</td>
<td>undef.</td>
<td>undef.</td>
<td>$\pi/2$</td>
<td></td>
</tr>
<tr>
<td>$\arctan x$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/4$</td>
<td>$\pi/3$</td>
<td>$\pi/2$</td>
<td>undef.</td>
<td>$\pi/3$</td>
<td></td>
</tr>
</tbody>
</table>

For negative values: $\arcsin(-x) = -\arcsin(x)$ and $\arctan(-x) = -\arctan(x)$. Blank entries in the table are not “nice” multiples of $\pi$.

Error bounds from the alternating series tests

If $\sum_{n=0}^{\infty} (-1)^n a_n$ converges by the alternating series test, then

$$|\text{error in the } M\text{th partial sum of the series}| \leq |\text{the } (M+1)\text{th term in the series}|$$
Section 1: Improper integrals, geometric sums and series

1. a. For which values of \( p \) does the integral \( \int_2^\infty \frac{1}{x^p} \, dx \) converge? Describe all such values.

   Because changing the lower endpoint from 1 to 2 (or the other way around) does not introduce any new issues, this integral converges exactly when
   \[
   \int_1^\infty \frac{1}{x^p} \, dx
   \]
   converges, so the correct answer is for all \( p > 1 \).

b. For which values of \( p \) does the integral \( \int_0^1 \frac{1}{x^p} \, dx \) converge? Describe all such values.

   For all \( p < 1 \).

c. For which values of \( a \) does the integral \( \int_{-\infty}^\infty e^{ax} \, dx \) converge? Describe all such values.

   Since the integral has problems at both endpoints, it needs to be split up:
   \[
   \int_{-\infty}^\infty e^{ax} \, dx = \int_{-\infty}^0 e^{ax} \, dx + \int_0^\infty e^{ax} \, dx
   \]
   – If \( a = 0 \), then \( e^{ax} = e^0 = 1 \), so both integrals are measuring an area of infinite length and constant height 1, so they both diverge. The whole integral consequently diverges.
   – If \( a > 0 \), then \( e^{ax} \to +\infty \) as \( a \to +\infty \), so \( \int_0^\infty e^{ax} \, dx \) diverges, and the whole integral consequently diverges.
   – If \( a < 0 \), then \( e^{ax} \to +\infty \) as \( a \to -\infty \), so \( \int_{-\infty}^0 e^{ax} \, dx \) diverges, and the whole integral consequently diverges.

   The correct answer: The integral converges for no values of \( a \).

2. In a.–i. below, fill in the blank with the correct asymptotic relation \( \prec \), \( \asymp \), or \( \succ \).

   a. \( n! + 2^n \prec n^n \)

      \( n! \) dominates every exponential growth function (\( n! \succ 2^n, 3^n, 4^n, \ldots \)), but is dominated by \( n^n \).

   b. \( x^5 + 5^x \succ (x^3 + 2^x)^2 \)

      The left-hand side is \( \asymp 5^x \) by the law of the dominant term. Expanding the right hand side, we have \( x^6 + 2 \cdot 2^x \cdot x^3 + (2^x)^2 \asymp 4^x \) by the law of the dominant term.

   c. \( e^n \asymp 3^n \)

   d. \( \ln x \prec x^{1/3} \)
e. \[ \frac{\sqrt{x^4 + x + 17}}{((2x)^6 + 35)^{1/3}} \asymp 55 \]

The left hand side is \( \asymp \frac{\sqrt{x^4}}{((2x)^6)^{1/3}} = \frac{x^2}{4x^2} = 1/4 \). All nonzero constants are \( \asymp \) to each other (they are the “stable” part of the spectrum, between decay and growth). In this case, \( 1/4 \asymp 55 \).

f. \( x^{-2} \gg 2^{-x} \)

Both functions decay, but \( x^{-2} \) decays slower. Another way to see this is \( x^2 \ll 2^x \), but taking reciprocals flips the asymptotic comparison relation.

g. \( \ln(x^{1/12}) \asymp \ln(x^{-1}) \)

Remember that \( \ln(x^a) = a \ln x \). So the left is \( \frac{1}{12} \ln x \) and the right is \( -\ln x \). The ratio of the two sides tends to a nonzero constant as \( x \to \infty \).

h. \( x \cdot \frac{1}{\ln(\ln(x))} \ll 1000000^{1000000} \cdot x \)

i. \( \arctan(x) \asymp \frac{1}{1/x + 1} \)

Both functions are “stable.” The left tends to \( \pi/2 \) and the right tends to 1 as \( x \to \infty \).

3. Circle all true statements:

i. \( \int_{-1}^{1} \frac{1}{x} \, dx = 0. \)

The integral must be broken up into \( \int_{-1}^{0} + \int_{0}^{1} \). Both of these integrals diverge (the right diverges by the \( p \)-test for \( \int_{0}^{1} \frac{1}{x^p} \, dx \)). The whole integral therefore diverges.

ii. \( \int_{-1}^{1} x^3 \, dx = 0. \)

This is a proper integral which evaluates to \( \frac{x^4}{4}\bigg|_{-1}^{1} = \frac{1}{4} - \frac{(-1)^4}{4} = 0 \). This is because the integrand is an odd function and the interval of integration is symmetric around 0.

iii. \( \int_{-1}^{1} \frac{1}{x^{1/3}} \, dx = 0. \)

This integral must be split at 0, but now both integrals converge. Since the function is odd and the interval of integration is symmetric around 0, the integral will converge to 0.

iv. None of the above are true.
4. Evaluate: \( \int_0^\infty \left( \frac{2x}{1+x^2} - \frac{1}{\frac{1}{2}x+1} \right) \, dx \).

Show all work and box your final answer. If you believe the integral diverges, say so.

As a quick check, the integrand is defined except at \( x = -2 \), which is not included in the interval of integration. We have

\[
\int \frac{2x}{1+x^2} \, dx = \ln|1 + x^2| + C \quad \text{and} \quad \int \frac{1}{\frac{1}{2}x+1} \, dx = 2 \ln|\frac{1}{2}x + 1| + C = \ln(|\frac{1}{2}x + 1|^2) + C
\]

by substitutions of \( u = 1 + x^2 \) and \( du = 2x \, dx \) in the first and \( u = \frac{1}{2}x + 1 \) and \( du = \frac{1}{2} \, dx \) in the second. Therefore,

\[
\int \left( \frac{2x}{1+x^2} - \frac{1}{\frac{1}{2}x+1} \right) \, dx = \ln \left| \frac{1 + x^2}{(\frac{1}{2}x + 1)^2} \right| + C
\]

At \( x = 0 \), the part before the +C is equal to \( \ln(1) = 0 \). As \( x \to \infty \), we have

\[
\lim_{x \to \infty} \ln \left| \frac{1 + x^2}{(\frac{1}{2}x + 1)^2} \right| = \ln \left| \lim_{x \to \infty} \left( \frac{1 + x^2}{\frac{1}{4}x^2 + x + 1} \right) \right| = \ln(4)
\]

So the answer is \( \ln(4) - \ln(1) = \ln(4) \).

5. Evaluate: \( \int_3^\infty \frac{1}{x^2 + 9} \, dx \).

Show all work and box your final answer. If you believe the integral diverges, say so.

Using the integral table entry on page 2, an antiderivative for the integrand is \( \frac{1}{3} \arctan\left( \frac{x}{3} \right) \). At \( x = 3 \)

this is \( \frac{1}{3} \arctan(1) = \frac{1}{3} \frac{\pi}{4} = \frac{\pi}{12} \). As \( x \to \infty \), this tends to \( \frac{1}{3} \frac{\pi}{2} = \frac{\pi}{6} \). The answer is therefore

\[
\frac{\pi}{6} - \frac{\pi}{12} = \frac{\pi}{12}.
\]
6. Consider the improper integral
\[ \int_{1/3}^{\infty} \frac{x}{\ln x \cdot \sqrt{(x-3)(x-5)(x-7)}} \, dx \]

a. Express this improper integral as a sum of simple improper integrals.

Remember that a “simple” improper integral is an improper integral that has only a single issue and that issue is located at one of the endpoints.

To simplify notation, you may use the abbreviation \( \int_{a}^{b} \) for \( \int_{a}^{b} \frac{x}{\ln x \cdot \sqrt{(x-3)(x-5)(x-7)}} \, dx \).

The integrand is undefined at \( x = 0, 1, 3, 5, 7 \), and we also have \( \infty \) as an upper endpoint.

Therefore, we split it as \( \int_{1/3}^{2} + \int_{2}^{4} + \int_{4}^{5} + \int_{5}^{6} + \int_{6}^{7} + \int_{7}^{8} + \int_{8}^{\infty} \). The \( * \) here (for instructive purposes only) marks where the single issue in each integral is.

b. Does this improper integral converge or diverge?

It diverges. We only need to check that one of the integrals from (a) diverges—the easiest to check is \( \int_{8}^{\infty} \), because it can be handled by limit comparison. The integrand is asymptotic to \( \frac{x}{\ln x \cdot \sqrt{x^3}} = \frac{1}{\ln x} \). Since \( \frac{1}{\ln x} \gg \frac{1}{x} \) and \( \int_{8}^{\infty} \frac{1}{x} \, dx \) diverges, \( \int_{8}^{\infty} \frac{1}{\ln x} \, dx \) diverges too.
7. Does the integral \( \int_0^\infty \frac{1 + \cos 3x}{e^x} \, dx \) converge or diverge? Justify your answer.

Converges by direct comparison. The numerator is \( 0 \leq 1 + \cos 3x \leq 2 \) and so the whole integrand satisfies \( 0 \leq \frac{1 + \cos 3x}{e^x} \leq \frac{2}{e^x} \). Since \( \int_0^\infty \frac{2}{e^x} \, dx \) converges, our integral does too.

8. Does the integral \( \int_0^1 \frac{1}{x^{3/2} \ln(1 + x) + \sqrt{x}} \, dx \) converge or diverge? Justify your answer.

Converges. Factoring the integrand yields \( \frac{1}{x^{1/2}(x \ln(x + 1) + 1)} \).

Consider the “bystander” term \( \frac{1}{x \ln(x + 1) + 1} \): \( x \) and \( \ln(x + 1) \) are increasing on \([0, 1]\) so the denominator of the bystander is increasing. Therefore, the bystander is decreasing, and so its min/max values are on the right/left endpoints of the interval \([0, 1]\). That is, \( 0 \leq \frac{1}{\ln(2) + 1} \leq \frac{1}{x \ln(x + 1) + 1} \leq 1 \). Our integrand is therefore between \( 0 \) and \( \frac{1}{\sqrt{x}} \), and since \( \int_0^1 \frac{1}{\sqrt{x}} \, dx \) converges, our integral does too.
9. Modern *humans* typically use base-10 *decimal* expansions to write numbers. That means that each digit corresponds to a multiple of a power of 10: for example, 

\[
132.37 = 100 + 30 + 2 + \frac{3}{10} + \frac{7}{100} = 1 \cdot 10^2 + 3 \cdot 10^1 + 2 \cdot 10^0 + 3 \cdot 10^{-1} + 7 \cdot 10^{-2}.
\]

*Computers*, on the other hand, typically use base-2 *binary* expansions. In binary, instead of using powers of 10, each digit corresponds to a power of 2, and the only digits are 0 and 1. So, for example, the decimal number 5.5 is written in binary as \(101.1\) because

\[
5.5 \text{ in decimal} = 5 + \frac{5}{10} = 4 + 0 + 1 + \frac{1}{2} = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 + 1 \cdot 2^{-1} = 101.1 \text{ in binary}.
\]

a. The binary number \(0.1010101\) is equal to

\[
\frac{1}{2} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{0}{2^4} + \frac{1}{2^5} + \frac{0}{2^6} + \frac{1}{2^7} + \frac{0}{2^8} + \frac{1}{2^9}
\]

Express the number above as a fraction in lowest terms. *Hint:* \(2^{10} = 1024\) and \(1023 = 3 \cdot 341\).

This is equal to \(\frac{1}{2}(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256}) = \frac{11}{2} \frac{1 - (1/4)^5}{1 - 1/4} = \frac{1}{2} \frac{1023/1024}{3/4} = \frac{1}{2} \frac{341}{256} = \frac{341}{512}\).

b. Write the binary number \(111.1\overline{1} = 111.111\ldots\) (the 1s repeat forever) as a whole number or as a fraction in lowest terms.

This is \(2^2 + 2^1 + 2^0 + 2^{-1} + 2^{-2} + 2^{-3} + \cdots\), so \(= 2^2(1 + 2^{-1} + 2^{-2} + \cdots) = 4 \sum_{n=0}^{\infty} (1/2)^n = 4 \cdot \frac{1}{1-1/2} = 8\).
Section 2: Sequences and series, convergence tests, power series

10. Suppose that $s$ is a real number. Consider the infinite series
\[
\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots .
\]

In the blanks below, describe all values of $s$ for which this series converges absolutely, converges conditionally, or diverges.

<table>
<thead>
<tr>
<th>CONVERGES ABS.</th>
<th>CONVERGES COND.</th>
<th>DIVERGES</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s &gt; 1$</td>
<td>(no values of $s$)</td>
<td>$s \leq 1$</td>
</tr>
</tbody>
</table>

11. Let $\{a_k\} = \{a_1, a_2, a_3, \ldots \}$ be a sequence, and let $\{S_n\}$ be the associated sequence of partial sums:
\[
S_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n
\]

(a) Circle the true statement(s) below.

i. If $\{a_k\}$ converges, then $\{|a_k|\}$ must also converge.
   Because if $\lim(a_k) = L$ then $\lim(|a_k|) = |L|$.

ii. If $\{|a_k|\}$ converges, then $\{a_k\}$ must also converge.
   False. If $a_k = (-1)^k$ then $\{|a_k|\}$ converges to 1 but $\{a_k\}$ does not converge.

iii. If $\{a_k\}$ converges, then $\{S_n\}$ must also converge.
    False. If $a_k = 1$ for all $k$, then $\{a_k\}$ converges to 1, but $S_n = n$, so $\{S_n\}$ tends to $+\infty$.

iv. If $\{S_n\}$ converges, then $\{a_k\}$ must also converge.
   ...to zero. This is the divergence test.

v. If $\{a_k\}$ is a sequence of positive numbers and $\{S_n\}$ is bounded, then $\{S_n\}$ must converge.
   If the terms of $\{a_k\}$ are all positive, then $\{S_n\}$ is increasing, hence monotone. A sequence that is monotone and bounded always converges—this is one of the two relationships between sequence properties you were expected to know.

vi. None of the above.
12. In this course we discussed in detail how to compute the error in partial sums of alternating series. But what about convergent series whose terms are all positive? One way to estimate the error in this case is to use the bounds from the integral test.

Suppose that $f$ is positive, decreasing, and continuous on $[1, \infty)$, that $\int_1^\infty f(x) \, dx$ converges. Then $\sum f(k)$ converges and we have for every integer $n \geq 1$ the following inequality:

$$
\int_n^\infty f(x) \, dx \leq \sum_{k=n}^\infty f(k) \leq f(n) + \int_n^\infty f(x) \, dx.
$$

a. Briefly explain how to justify the inequality on the left above (drawing a picture might help!).

This is essentially because $f(x)$ is decreasing on $[1, \infty)$ and therefore on $[n, \infty)$, left-handed Riemann sums of $f(x)$ on $[n, \infty)$ are overestimates. The series in the middle is the left-handed Riemann sum of $\int_n^\infty f(x) \, dx$ where the subintervals all have width of 1 unit. (The picture is better at explaining this, so draw it.)

b. Let $S = \sum_{k=1}^{n-1} f(k)$. Briefly explain how we can use the inequalities above to conclude that

$$
S + \underbrace{\int_n^\infty f(x) \, dx}_{A_n} \leq \sum_{k=1}^\infty f(k) \leq S + f(n) + \underbrace{\int_n^\infty f(x) \, dx}_{B_n}.
$$

We just add $S$ to all three parts of the inequality before (a). The only big change is that $S + \sum_{k=n}^\infty f(k) = \sum_{k=1}^{n-1} f(k) + \sum_{k=n}^\infty f(k) = \sum_{k=1}^\infty f(k)$.

c. With $A_n$ and $B_n$ as defined in (b), briefly explain why we know that $B_n - A_n \to 0$ as $n \to \infty$.

Note: This means that the range of values $[A_n, B_n]$ where $\sum_{k=1}^\infty f(k)$ lies gets smaller and smaller as $n \to \infty$. This allows us to estimate $\sum_{k=1}^\infty f(k)$ to any desired precision.

We have $B_n - A_n = f(n)$. Since $\sum f(k)$ converges, $\lim_{n \to \infty} f(n) = 0$ by the divergence test. So $\lim_{n \to \infty} (B_n - A_n) = 0$. 

13. Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and $G(x) = \sum_{n=0}^{\infty} b_n x^n$ with all $a_n > 0$ and all $b_n > 0$. Suppose that 
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2} \quad \text{and} \quad \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = 1
\]

Let $x_0$ be a real number. Circle all of the following statements that are true:

i. If $x_0 < 0$, then the series $F(x_0)$ is alternating.

ii. If $F(x)$ converges at $x = x_0$, then $G(x)$ converges at $x = x_0$ too.

iii. If $G(x)$ converges at $x = x_0$, then $F(x)$ converges at $x = x_0$ too.

   For (ii.) and (iii.): The calculations above tell us that for $F(x)$, 
   \[
   L(x) = \lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{2}
   \]
   so $F$ converges on the interval $(-2, 2)$ by the ratio test. (It may converge or diverge at the endpoints—there is no way to determine this with the info given.) On the other hand, $G$ converges on the interval $(-1, 1)$ (and possibly at the endpoints).

Hence, (ii) is false (since if $x_0 = 3/2$, $F$ converges at $x_0$ but $G$ does not). (iii) is true because if $G$ converges at $x_0$ then $-2 < -1 \leq x_0 \leq 1 < 2$, so $x_0$ is in $F$’s IoC.

iv. The power series $F(x) + G(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$ converges at $x = 1/3$.

   True because $1/3$ is in both $F$’s and $G$’s IoCs. Thus $F(1/3)$ and $G(1/3)$ both converge, and so their sum converges too.

v. The series $F(-1)$ converges absolutely.

   True. Even though the series is alternating, it still converges absolutely, because the ratio test will yield $L = 1/2$ when applied to the series $F(-1) = \sum (-1)^n a_n$.

vi. None of the above.

14. Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges conditionally.

   a. Explain briefly what this means.

      That $\sum a_n$ converges but $\sum |a_n|$ diverges.

   b. Again: Suppose that $\sum_{n=1}^{\infty} a_n$ converges conditionally.

      Using this information, decide in (i–vii) whether each of the given statements is ALWAYS TRUE, SOMETIMES (but not always) TRUE, or ALWAYS FALSE and circle your answers.
i. The sequence \( \{|a_n|\} \) converges.

<table>
<thead>
<tr>
<th>ALWAYS TRUE</th>
<th>SOMETIMES TRUE</th>
<th>ALWAYS FALSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>...to zero. Because ( \lim_{n \to 0} (a_n) = 0 ) (this is the divergence test, yet again) so ( \lim_{n \to 0} (</td>
<td>a_n</td>
<td>) =</td>
</tr>
</tbody>
</table>

ii. The series \( \sum_{n=1}^{\infty} (a_n + 1) \) converges.

<table>
<thead>
<tr>
<th>ALWAYS TRUE</th>
<th>SOMETIMES TRUE</th>
<th>ALWAYS FALSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>By the divergence test again: ( \lim_{n \to \infty} \sum (a_n + 1) = \lim_{n \to \infty} (a_n) + 1 = 0 + 1 = 1 \neq 0. )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

iii. The series \( \sum_{n=1}^{\infty} a_n \) is an alternating series.

<table>
<thead>
<tr>
<th>ALWAYS TRUE</th>
<th>SOMETIMES TRUE</th>
<th>ALWAYS FALSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>There is nothing in the definition of “conditionally convergent” that requires the series to be alternating. The sign pattern of terms could be ( ++--+++--), for instance, or it could be completely random.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

iv. The sequence \( \{a_n\} \) has infinitely many negative terms.

<table>
<thead>
<tr>
<th>ALWAYS TRUE</th>
<th>SOMETIMES TRUE</th>
<th>ALWAYS FALSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>If a series has only finitely many negative terms, then for some sufficiently large ( N ), it is equal to (a finite sum) + ( \sum_{n=N}^{\infty} a_n ) where the second sum includes no negative terms. Since ( \sum_{n=1}^{\infty} a_n ) converges, ( \sum_{n=N}^{\infty} a_n ) converges (since we can change the lower endpoint without affecting convergence), and because it only has positive terms, ( \sum_{n=N}^{\infty} a_n ) converges absolutely. It follows that ( \sum_{n=1}^{\infty} a_n ) also converges absolutely, since adding finitely many terms to a series does not affect its convergence behavior.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

v. The sequence \( \{a_n\} \) satisfies \( |a_n| \asymp \frac{1}{n} \).

<table>
<thead>
<tr>
<th>ALWAYS TRUE</th>
<th>SOMETIMES TRUE</th>
<th>ALWAYS FALSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sometimes true because ( \sum \frac{(-1)^n}{a_n} ) converges conditionally, but ( \sum \frac{(-1)^n}{\sqrt{n}} ) also converges conditionally and ( \frac{1}{\sqrt{n}} \neq \frac{1}{n}. )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

vi. The sequence \( \{a_n\} \) satisfies \( |a_n| \asymp \frac{1}{n^2} \).

<table>
<thead>
<tr>
<th>ALWAYS TRUE</th>
<th>SOMETIMES TRUE</th>
<th>ALWAYS FALSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Because limit comparison would then say that ( \sum</td>
<td>a_n</td>
<td>) converges, and then ( \sum a_n ) would converge absolutely and not conditionally.</td>
</tr>
</tbody>
</table>
15. For the series (a–h) below, determine whether the series **CONVERGES** or **DIVERGES** and circle your answer. If an extra space is provided for $L$, fill it in with the value you would find if you applied the ratio test (regardless of whether that is a “good” test to apply for that series).

You do not need to show your work or specify absolute vs. conditional convergence.

The next page and a half have been left blank for extra work. It will not be graded.

| a. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ | b. $\sum_{n=1}^{\infty} \frac{\sqrt{n^4 + 4^n}}{n^4 + n + 1}$ |
| CONVERGES | DIVERGES |
| By integral test. | By divergence test. |
| c. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n + 1)^{1/3}}$ | d. $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}$ |
| CONVERGES | DIVERGES |
| By alternating series test. | By ratio test. |
| $L = 1$ | $L = 4$ |
| e. $\sum_{n=0}^{\infty} (\cos 1)^n$ | f. $\sum_{n=0}^{\infty} 2^n \sin(\pi n)$ |
| CONVERGES | DIVERGES |
| $-1 < \cos(1) < 1$, so series is geometric with $|r| < 1$. | All terms of series are zero. |
| g. $\sum_{n=0}^{\infty} \frac{n^{1000}}{1.0001^n}$ | h. $\sum_{n=0}^{\infty} \frac{n!}{(n + 2)!}$ |
| CONVERGES | DIVERGES |
| Ratio test. | Limit comparison: $\frac{n!}{(n+2)(n+1)} \propto \frac{1}{n^2}$. |
| $L = \frac{1}{1.0001}$ | $L = 1$ |
Left blank for additional work on problem 6.
16. a. Write down a power series that represents (converges to) either $\ln x$ or $\ln(x + 1)$ on its interval of convergence (be sure to specify which of $\ln x$ or $\ln(x + 1)$ you chose).

$$\ln(x + 1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1}.$$ 

b. What is the interval of convergence of the power series you wrote down above?

It converges on $(-1, 1]$. 

Left blank for additional work on problem 6.
Section 3: Taylor series and applications

17. Below, solve for $X$, $Y$, and $Z$, or state that no solutions exist. It is possible that multiple solutions exist, in which case you only have to specify one solution for full credit.

   a. Solve $\sum_{n=0}^{\infty} \frac{3^n X^n}{n!} = 5$ for $X$ or state that no solutions exist.

      This is $e^{3X} = 5$, so $X = \frac{1}{3} \ln 5$.

   b. Solve $\sum_{n=0}^{\infty} Y^n = \frac{1}{2}$ for $Y$ or state that no solutions exist.

      The left-hand side converges to $\frac{1}{1-Y}$ on $(-1, 1)$. Solving $\frac{1}{1-Y} = \frac{1}{2}$ yields $Y = -1$, but this is outside the interval of convergence. There are no solutions.

   c. Solve $\sum_{n=0}^{\infty} \frac{(-1)^n \pi 2n}{4^n (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n Z^{2n+1}}{(2n+1)!}$ for $Z$ or state that no solutions exist.

      The left is $\cos(\pi/2) = 0$ while the right is $\sin(Z)$. $Z = 0$ is one of infinitely many solutions ($Z = k\pi$ for any integer $k$ is a solution).
18. Evaluate the limit \( \lim_{x \to 0} \frac{(\sin x)(\arctan x)(1 - \sqrt{1 + x})^2}{x^2(\cos x - 1)} \).

You may use whatever method you like, but you must show your work.

Draw a box around your final answer.

We replace each factor with a Taylor polynomial for that factor (at \( x = 0 \)) having at least one nonzero term: \( \sin x \approx x \), \( \arctan x \approx x \),

\[
1 - \sqrt{1 + x} = 1 - \left[ 1 + \left( \frac{1/2}{1/2} \right) x + \left( \frac{1/2}{2} \right) x^2 + \cdots \right] \approx -\frac{1}{2} x
\]

and \( 1 - \cos x = 1 - (1 - \frac{1}{2} x^2 + \cdots) \approx \frac{1}{2} x^2 \). All these \( \approx \) are contingent on the condition “when \( x \) is close to zero.” Our limit is therefore

\[
\lim_{x \to 0} \frac{x \cdot x \cdot (-x/2)^2}{x^2(x^2/2)} = \frac{1/4}{1/2} = 1/2
\]
19. a. Evaluate the integral \( \int_0^1 \frac{1 - \cos(x^3)}{x} \, dx \) in terms of a convergent infinite series.

Your answer should be presented in \( \sum \) form. You do not need to prove that your series converges.

We have

\[
\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots
\]

so

\[
1 - \cos(x^3) = -\frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}
\]

and

\[
\frac{1 - \cos(x^3)}{x} = -\frac{x^5}{2!} + \frac{x^{11}}{4!} - \frac{x^{17}}{6!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n x^{6n-1}}{(2n)!}
\]

We are now ready to integrate:

\[
\int_0^1 \frac{1 - \cos(x^3)}{x} \, dx = \int_0^1 \sum_{n=1}^{\infty} \frac{(-1)^n x^{6n-1}}{(2n)!} \, dx
\]

\[
= \sum_{n=1}^{\infty} \int_0^1 \frac{(-1)^n x^{6n-1}}{(2n)!} \, dx
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \int_0^1 x^{6n-1} \, dx
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{1}{6n}
\]

b. Estimate the value of the integral in (a) so that the error in your estimate is \( < 10^{-3} \).

Your answer should be a finite sum/difference of fractions. It may not include \( \sum \) or \( \cdots \).

Draw a box around your final answer.

The series above is \( \frac{1}{2! \cdot 6} - \frac{1}{4! \cdot 12} + \frac{1}{6! \cdot 18} - \frac{1}{8! \cdot 24} + \cdots \). We have 6! = 720, and so 6! \cdot 18 is clearly larger than \( 10^3 = 1000 \). Thus, the error in the estimate

\[
\frac{1}{2! \cdot 6} - \frac{1}{4! \cdot 12} = \frac{23}{288} \approx 0.079861
\]

is thus \( \leq \frac{1}{6! \cdot 18} < 10^{-3} \). Indeed, the “true value” of the integral is \( \approx 0.079937 \).
20. Suppose that the power series \( y = F(x) = \sum_{n=0}^{\infty} c_n x^n \) satisfies the differential equation
\[
y'' = 4y + (x + 1)^2 \quad \text{with} \quad y(0) = -1 \quad \text{and} \quad y'(0) = 2.
\]
What is the value of \( c_3 \), the coefficient of \( x^3 \), in \( F(x) \)? Draw a box around your final answer.

We need to figure out the third derivative of \( y \) at 0. Differentiating the differential equation,
\[
y''' = 4y' + 2(x + 1)
\]
and plugging in \( x = 0 \) on the right gives us \( 4y'(0) + 2(0 + 1) = 8 + 2 = 10 \). Thus \( y'''(0) = 10 \). This means that \( F'''(0) = 10 \), and \( c_3 = \frac{F'''(0)}{3!} \) by the relationship between a power series’ coefficients and its derivatives at the center. So! \( c_3 = \frac{10}{3!} = \frac{10}{6} = \frac{5}{3} \).
Breadth questions!

Circle the correct answers. Four out of five must be answered correctly for full credit.

a. The gravitational interactions of three or more objects always display periodic behavior.

TRUE        FALSE

b. Which Greek letter is used to denote the limit of \[ \sum_{k=1}^{n} \frac{1}{k} - \ln n \] as \( n \to \infty \)?

\( \pi \)        \( \gamma \)        \( \Omega \)

c. What phenomenon causes ringing artifacts in images and can lead to medical misdiagnoses?

Gibbs Phenomenon        Ultraviolet Catastrophe        Poisson Distribution

d. To what number does the ratio of consecutive Fibonacci numbers tend?

\[ \frac{\pi^2}{6} \]        \[ \frac{1 - \sqrt{5}}{2} \]        \[ \frac{1 + \sqrt{5}}{2} \]

e. What is the name of the function \( f(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) defined for \( s > 1 \)?

harmonic series        Riemann zeta        Bessel function

The remaining space is blank for extra work.