Failure to follow the instructions below will constitute a breach of the Stanford Honor Code:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone or any other electronics during the exam for any reason.
- You must sit in your assigned seat.
- You may not communicate with anyone other than the course staff during the exam, or look at anyone else’s solutions.
- Additionally, you may not discuss or communicate directly or indirectly the contents of this exam with ANYONE other than the course staff until tomorrow, January 26th.
- You have 80 minutes to complete this exam.

I will not discuss this exam with ANYONE other than the course staff until: ________________
I understand and accept these instructions.

Signature: _______________________________________________________

Remember to show your work and justify your answer if required (additional tips are on the next page). Present all solutions in as organized a manner as possible.

GOOD LUCK!
Here are some tips:

- If you have time, it’s always a good idea to check your work.

- If you get the wrong answer for an integral but show your work, chances are good that we can award you partial credit.

- **DO NOT** attempt to estimate any of your answers as decimals. For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is exact.

- The boxes at the end of each topic are for grading purposes only. Do not touch or look at these boxes. Pretend they are not there.

- The last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

### Integral table entries you may need

- $\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \frac{u - a}{u + a} + C$
- $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C$
- $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \left( \frac{u}{a} \right) + C$

### Inverse trigonometric functions

The table below gives important values of the three main inverse trig functions.

| $x$ | 0 | $\frac{1}{\pi}$ | $\frac{1}{\sqrt{3}}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\sqrt{3}$ | $\infty$
|-----|---|----------------|----------------------|----------------------|---|----------|----------|
| $\arcsin x$ | 0 | $\pi/6$ | $\pi/3$ | $\pi/4$ | $\pi/3$ | $\pi/2$ | $\infty$
| $\arccos x$ | $\pi/2$ | $\pi/2$ | $\pi/4$ | $\pi/4$ | $\pi/4$ | $\pi/3$ | $\pi/2$
| $\arctan x$ | 0 | $\pi/6$ | $\pi/4$ | $\pi/4$ | $\pi/4$ | $\pi/3$ | $\pi/2$

Blank entries are either undefined or are not “nice” multiples of $\pi$.

Also: $\arcsin(-x) = -\arcsin(x)$, $\arccos(-x) = \pi - \arccos(x)$, and $\arctan(-x) = -\arctan(x)$.

REMEMBER that $\sin^n x$ means $(\sin x)^n$. 
1. Which of the following integrals is improper? Circle all improper integrals.

\[
\int_0^1 \frac{dx}{\sqrt{x + 1}} \quad \int_0^1 \frac{dx}{\sqrt{x - 1}} \quad \int_0^\infty \frac{dx}{\sqrt{x + 1}} \quad \int_1^2 \frac{dx}{e^{x^2}} \quad \int_{-1}^1 e^{-x^2} \, dx
\]

2. Rewrite the following improper integrals as a sum of simple improper integrals (remember that a “simple” improper integral is one that has only a single issue and that issue occurs at an endpoint). If the integral is already simple, say so.

a. \( \int_{-2}^2 \frac{x + 1}{x - 1} \, dx \)

b. \( \int_{-1}^1 \ln |x(x - 1)| \, dx \)

c. \( \int_0^\infty \frac{e^{-x}}{x^2 + \sqrt{x}} \, dx \)

d. \( \int_0^\pi \frac{1}{\cos x} \, dx \)

e. \( \int_0^1 \frac{1}{1 - \sqrt{x}} \, dx \)

3. In HW1 you learned about the \( \Gamma \)-function, defined by the formula

\[
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt
\]

and you proved that \( \Gamma(x + 1) = x \Gamma(x) \).

a. What method of integration allowed you to prove the identity \( \Gamma(x + 1) = x \Gamma(x) \)? Name the method.

b. If \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \), then what is \( \Gamma\left(\frac{5}{2}\right) \) equal to?

4. Fill in the blanks below with the correct choice of \(<, \infty, >\) (as \( x \to \infty \) or \( n \to \infty \)):

a. \( e^x \underline{\quad} \frac{2^x}{\infty} \)

b. \( e^{-x} \underline{\quad} \frac{(1/3)^x}{<} \)

c. \( 1 - x - 4x^2 \underline{\quad} \frac{\sqrt{x^4 + 1}}{>} \)

d. \( x(\ln x)^2 \underline{\quad} \frac{x^2 \ln x}{>} \)

e. \( n! \underline{\quad} \frac{(2^n)^2}{\infty} \)

f. \( \frac{n}{n^2(2n+1)^2} \underline{\quad} \frac{1}{\searrow} \)

g. \( \ln\left(\frac{1}{x}\right) \underline{\quad} \ln\left(\frac{1}{\sqrt{x}}\right) \)

5. True or false: The integral \( \int_1^\infty \frac{1}{x^p} \, dx \) converges as long as \( p \geq 1 \).

6. Partario is attempting to determine if the improper integral

\[
\int_0^1 \frac{1}{x + \sqrt{x}} \, dx
\]

converges or diverges.
a. Partario’s argument is as follows:

By the law of the dominant term, \( x + \sqrt{x} \propto x \) (as \( x \to \infty \)). Thus, \( \frac{1}{x + \sqrt{x}} \propto \frac{1}{x} \) (as \( x \to \infty \)).

Since \( \int_0^1 \frac{1}{x} \, dx \) diverges (because \( \int_0^1 \frac{1}{x^p} \, dx \) diverges if \( p \geq 1 \)), the improper integral in (*) must also diverge.

Whether the integral converges or diverges, Partario’s argument is spurious. Explain briefly why his argument does not work.

b. Does the integral above converge or diverge? Justify your answer.

7. What is \( 1 + 3 + 3^2 + 3^3 + \cdots + 3^9 \) equal to? You may leave your answer unsimplified.

8. Evaluate the improper integral \( \int_2^\infty e^{-x} \, dx \). Show all steps and draw a box around your answer. If you discover that the integral diverges, your boxed answer should be DIVERGES.

9. Evaluate the improper integral \( \int_{-\infty}^\infty \frac{dx}{4x^2 + 9} \). Show all steps and draw a box around your answer. If you discover that the integral diverges, your boxed answer should be DIVERGES.

10. Evaluate the improper integral \( \int_0^4 \frac{1}{2\sqrt{x}} \, dx \). Show all steps and draw a box around your answer. If you discover that the integral diverges, your boxed answer should be DIVERGES.

11. Evaluate the improper integral \( \int_{-1}^1 \frac{1}{\sqrt{1 - x^2}} \, dx \). Show all steps and draw a box around your answer. If you discover that the integral diverges, your boxed answer should be DIVERGES.

12. Does the integral \( \int_1^\infty \sqrt{\frac{9x^4}{x^6 + 2x + 5}} \, dx \) converge or diverge? Justify your answer.

13. Does the integral \( \int_0^\pi \frac{(\sin \theta + 1)^2}{\theta^3} \, d\theta \) converge or diverge? Justify your answer.

14. Determine whether the following infinite series converge or diverge. Justify your answer. If the series converges and you know the number to which it converges, write it down.

a. \( \sum_{k=0}^{\infty} 3 \cdot (-1)^k 4^{-k} \)

b. \( \sum_{n=0}^{\infty} \sqrt{2^n} \)

c. \( \sum_{n=0}^{\infty} \frac{n}{2n + 1} \)

d. \( \sum_{n=1}^{\infty} \frac{1}{n^3} \)