Math 21, Winter 2017 — Schaeffer
Midterm Exam 1 (February 1st, 2017)

Failure to follow the instructions below will constitute a breach of the Stanford Honor Code:

• You may not use a calculator or any notes or book during the exam.
• You may not access your cell phone or any other electronics during the exam for any reason.
• You must sit in your assigned seat.
• You may not communicate with anyone other than the course staff during the exam, or look at anyone else’s solutions.
• Additionally, you may not discuss the contents of this exam with ANYONE other than the course staff until 9:00 tonight, February 1st.
• You have 80 minutes to complete this exam. If the course staff must ask you to stop writing or to turn in your exam more than twice after time is called, you will receive a score of zero.

I understand and accept these instructions.

Signature: _______________________________________________________

Remember to show your work and justify your answer if required (additional tips are on the next page). Present all solutions in as organized a manner as possible.

GOOD LUCK!
Here are some tips:

- If you have time, it’s always a good idea to check your work.
- If you get the wrong answer but show your work, you have a better chance of receiving partial credit.
- **DO NOT** attempt to estimate any of your answers as decimals. For example, \(1 - \frac{1}{\pi}\) is a much better answer than 0.682, because it is exact.
- The very last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

**Integral table entries you may need**

\[
\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C
\]

\[
\int \frac{du}{(u - r)(u - s)} = \frac{1}{r - s} \ln \left| \frac{u - r}{u - s} \right| + C
\]

\[
\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C
\]

\[
\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \left( \frac{u}{a} \right) + C
\]

In the entries above, \(a, r, s\) are constants such that \(a \neq 0\) and \(r \neq s\).

**Values of arcsine and arctangent**

The table below gives important values of the arcsine and arctangent functions:

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>(\frac{1}{2})</th>
<th>(\frac{1}{\sqrt{3}})</th>
<th>(\frac{\sqrt{3}}{2})</th>
<th>1</th>
<th>(\sqrt{3})</th>
<th>(\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{arcsin} \ x)</td>
<td>0</td>
<td>(\frac{\pi}{6})</td>
<td>(\frac{\pi}{4})</td>
<td>(\frac{\pi}{3})</td>
<td>(\frac{\pi}{2})</td>
<td>undef.</td>
<td>undef.</td>
</tr>
<tr>
<td>(\text{arctan} \ x)</td>
<td>0</td>
<td>(\frac{\pi}{6})</td>
<td>(\frac{\pi}{4})</td>
<td>(\frac{\pi}{3})</td>
<td>(\frac{\pi}{4})</td>
<td>(\frac{\pi}{3})</td>
<td>(\frac{\pi}{2})</td>
</tr>
</tbody>
</table>

For negative values: \(\text{arcsin}(-x) = -\text{arcsin}(x)\) and \(\text{arctan}(-x) = -\text{arctan}(x)\). Blank entries in the table are not “nice” multiples of \(\pi\).
In Problems 1–9 justification of your answer is not required for full credit.

Some problems differ between forms. The short answer problems (1–9) were not substantively different, so I have included only solutions for one form. Problems 4 and 16 were identical on both forms.

1. For which values of $p$ does the integral $\int_1^\infty \frac{1}{x^p} \, dx$ converge? For which values of $p$ does it diverge?

   It converges for $p > 1$ and diverges for $p \leq 1$.

2. For which values of $p$ does the integral $\int_0^1 \frac{1}{x^p} \, dx$ converge? For which values of $p$ does it diverge?

   It converges for $p < 1$ and diverges for $p \geq 1$.

3. Express the following improper integral as a sum of simple* improper integrals: $\int_{-1}^{\infty} \frac{dx}{x(x-2)}$.

   * A “simple” improper $\int$ is one that has only a single issue, and that issue occurs at an endpoint.

   In the interests of time you may abbreviate “$\int_a^b \frac{dx}{x(x-2)}$” as “$\int_a^b$” in your answer.

   $\int_{-1}^0 + \int_0^1 + \int_1^2 + \int_2^3 + \int_3^\infty$

4. Partario is attempting to evaluate the integral

   $\int_{-\pi/2}^{\pi/2} \tan \theta \, d\theta$

   whose integrand has asymptotes at both endpoints of integration.

   Using the (correct) integration formula $\int \tan \theta \, d\theta = -\ln |\cos \theta| + C$, Partario (correctly) obtains

   $\int_{-a}^{a} \tan \theta \, d\theta = 0$

   whenever $0 \leq a < \frac{\pi}{2}$. Taking the limit of the above as $a \to \left(\frac{\pi}{2}\right)^-$ he concludes $\int_{-\pi/2}^{\pi/2} \tan \theta \, d\theta = 0$.

   Is Partario’s conclusion correct? If so, write PARTARIO IS CORRECT. If you believe it to be incorrect, then what is the correct conclusion? You do not need to justify your answer.

   Draw a box around your final answer.

   Partario is incorrect: The integral diverges.

   The integral he’s looking at is not simple. Splitting it into simple improper integrals: $\int_{-\pi/2}^{0} + \int_{0}^{\pi/2}$.

   Next we evaluate one of these simple integrals, per usual:

   $\int_{0}^{\pi/2} \tan \theta \, d\theta = \lim_{b \to (\pi/2)^-} \left[ -\ln |\cos b| + \ln |\cos 0| \right] = \infty + 0$

   so $\int_{0}^{\pi/2} \tan \theta \, d\theta$ diverges. Thus, the whole integral diverges.
Note: I’m writing up a short vignette that explains more in depth why it is reasonable to conclude that this integral diverges, even though it seems like the answer intuitively should be zero, from Partario’s argument and from looking at a graph of \( y = \tan x \). There are justifications that are better than “Because this is the way we were taught how to evaluate improper integrals in calculus class.”

5. In a–g fill in the blank with the correct asymptotic relation symbol: \(<\), \(\leq\), or \(\geq\) (as \( x \to \infty \)).

   a. \( 4x^3 \ll (\frac{x^4}{x^2 + 1})^2 \)

   b. \( e^{-x} \ll (1/2)^x \)

   c. \( x^2 + e^x \ll x^3 + e^x \)

   d. \( (1 - \sqrt{x})^5 \ll \frac{x^5}{1} \)

   e. \( x(\ln x)^2 \ll x^2 \ln x \)

   f. \( \ln(3^x) \asymp \ln(5^x) \)

   g. \( x^3 + e^{-x} \gg x^2 + e^{-x} \)

6. Which of the following statements about sequences and series are true? Circle all true statements.

   i. All convergent sequences are monotone. \textbf{False}.

   ii. All divergent sequences are unbounded. \textbf{False}.

   iii. Any increasing sequence that is bounded (above and below) must converge. \textbf{True}.

   iv. If the sequence \( a_1, a_2, a_3, \ldots \) consists of positive terms, then the corresponding sequence of partial sums \( a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots \) is increasing. \textbf{True}.

   v. If the sequence \( a_1, a_2, a_3, \ldots \) converges, then the corresponding infinite series \( a_1 + a_2 + a_3 + \cdots \) must also converge. \textbf{False}.

7. Write down an example of a sequence that is bounded but not convergent.

\[ (-1)^n \]

8. What is \( 6 - 6^2 + 6^3 - 6^4 + \cdots + 6^{27} \) equal to? You may express your answer as an unsimplified arithmetic expression. However, your answer may not include “\( \sum \)” or “\( \cdots \)”.

Draw a box around your final answer.

\[
6 - 6^2 + 6^3 - 6^4 + \cdots + 6^{27} = 6(1 - 6 + 6^2 - 6^3 + \cdots + 6^{26}) = 6 \sum_{k=0}^{26} (-6)^k = 6 \cdot \frac{(-6)^{27} - 1}{(-6) - 1}
\]

The answer above is fine, but if you simplify, you’ll get 877277459209259356746.
9. Evaluate the infinite geometric series $9 + \frac{9}{4} + \frac{9}{16} + \frac{9}{64} + \cdots$. If you believe it diverges, say so.

**Draw a box around your final answer.**

$$9 + \frac{9}{4} + \frac{9}{16} + \frac{9}{64} + \cdots = 9(1 + \frac{1}{4} + \frac{1}{16} + \cdots) = 9 \sum_{k=0}^{\infty} (\frac{1}{4})^k = \frac{9}{1 - \frac{1}{4}} = 12$$

In Problems 10–13, evaluate the improper integral. **Show your work.** If you believe the integral to diverge, write DIVERGES as your final answer.

Remember that on the second page of the exam there are **integration table entries** that might apply, and there is a **table of values of arcsine and arctangent** as well. For full credit, your final answers should not include unresolved “arcsin” or “arctan”.

**Draw boxes around your final answers.**

10 \[ \int_0^8 \frac{dx}{x^{2/3}} \]

\[
\int_0^8 \frac{dx}{x^{2/3}} = \lim_{a \to 0^+} \int_a^8 \frac{dx}{x^{2/3}} \\
= \lim_{a \to 0^+} \left[ 3x^{1/3} \right]^8_a \\
= \lim_{a \to 0^+} \left[ 3 \cdot 8^{1/3} - 3a^{1/3} \right] \\
= 3 \cdot 8^{1/3} \\
= 6 \]
\[ \int_2^{\infty} xe^{-x^2} \, dx = \int_2^{\infty} xe^{-x^2} \, dx = \lim_{b \to \infty} \int_2^b xe^{-x^2} \, dx = \lim_{b \to \infty} \frac{1}{2} \int_4^{b^2} e^u \, du \quad u = x^2, \, du = 2x \, dx = \lim_{b \to \infty} \left[ -\frac{e^{-u}}{2} \right]_4^{b^2} = \lim_{b \to \infty} \left[ -\frac{e^{-b^2}}{2} + \frac{e^{-4}}{2} \right] = 0 + \frac{e^{-4}}{2} = \frac{1}{2e^4} \]
\[ \int_0^\infty \frac{dx}{9x^2 + 25} \]

\[ \int \frac{dx}{9x^2 + 25} = \int \frac{dx}{(3x)^2 + 5^2} \]

\[ = \frac{1}{3} \int \frac{du}{u^2 + a^2} \quad u = 3x, \ du = 3 \, dx, \ a = 5 \]

\[ = \frac{1}{15} \arctan\left(\frac{3x}{5}\right) + C \]

\[ \int_0^\infty \frac{dx}{9x^2 + 25} = \lim_{a \to \infty} \int_a^0 \frac{dx}{9x^2 + 25} \]

\[ = \lim_{a \to \infty} \left[ \frac{1}{15} \arctan\left(\frac{3x}{5}\right) \right]_a^0 \]

\[ = \lim_{a \to \infty} \left[ \frac{1}{15} \arctan(0) - \frac{1}{15} \arctan\left(\frac{3a}{5}\right) \right] \]

\[ = 0 - \frac{1}{15} \cdot \left( -\frac{\pi}{2} \right) \]

\[ = \frac{\pi}{30} \]
\[ \int_0^1 \frac{e^x}{\sqrt{e^x - 1}} \, dx \]

\[
\int_0^1 \frac{e^x}{\sqrt{e^x - 1}} \, dx = \lim_{a \to 0^+} \int_a^1 \frac{e^x}{\sqrt{e^x - 1}} \, dx \\
= \lim_{a \to 0^+} \int_{e^a - 1}^{e-1} \frac{du}{\sqrt{u}} \\
= \lim_{a \to 0^+} \left[ 2\sqrt{u} \right]_{e^a - 1}^{e-1} \\
= \lim_{a \to 0^+} \left[ 2\sqrt{e - 1} - 2\sqrt{e^a - 1} \right] \\
= 2\sqrt{e - 1} - 2\sqrt{1 - 1} \\
= 2\sqrt{e - 1}
\]
In Problems 14–16, determine if the improper integral converges or diverges using the method of your choice. If you evaluate the improper integral, show your work. If you use comparison (direct or asymptotic), outline your argument well enough that we can understand how you came to your conclusion.

You may use the convergence conditions for $\int_1^\infty \frac{1}{x^p} \, dx$, $\int_1^\infty \frac{1}{x^p} \, dx$, and $\int_0^\infty r^{-x} \, dx$ without justification, but mention those conditions if you use them.

Draw boxes around your final answers.

14. $\int_1^\infty \frac{x}{(x + 1)(4x^2 + 1)} \, dx$

   The integral converges. The integrand is asymptotically equivalent to $\frac{1}{x^3}$, and $\int_1^\infty \frac{1}{x^3} \, dx$ converges.

15. $\int_0^\infty \frac{e^{-x}}{x^2 + 1} \, dx$

   The integral converges. Here are two ways to justify this:
   - (Asymptotic comparison.) We have $\frac{e^{-x}}{x^2 + 1} < e^{-x}$ and $\int_0^\infty e^{-x} \, dx$ converges.
   - (Direct comparison.) We have $0 \leq e^{-x} \leq 1$ whenever $x \geq 0$. Thus,
     $$0 \leq \frac{e^{-x}}{x^2 + 1} \leq \frac{1}{x^2 + 1}$$

   and since $\int_0^\infty \frac{dx}{x^2 + 1}$ converges (to $\frac{\pi}{2}$), the integral in the problem converges.

16. $\int_0^{\pi/2} \frac{dx}{x + \sin x}$

   Hint: $0 \leq \sin x \leq x$ when $0 \leq x \leq \frac{\pi}{2}$.

   The integral diverges. We have $0 \leq \sin x \leq x$ when $0 \leq x \leq \frac{\pi}{2}$ (by the hint!), so $x \leq x + \sin x \leq 2x$ for those values of $x$. Taking reciprocals yields

   $$0 \leq \frac{1}{2x} \leq \frac{1}{x + \sin x} \leq \frac{1}{x}$$

   So, since $\int_0^{\pi/2} \frac{1}{2x} \, dx$ diverges, the original integral does as well.