Failure to follow the instructions below is a breach of the Stanford Honor Code:

- You may not use or consult any book or notes during the exam.*
- You may not use a calculator or the calculator function on any electronic device during the exam.*
- You may not access any internet-capable electronic device during the exam,* including smartphones and smartwatches, for any reason. These devices must be switched to “airplane mode” and disconnected from all wireless networks (both cellular and wifi) during the exam*.
- You must sit in your assigned seat.
- You may not communicate with anyone other than the course staff during the exam,* or look at anyone else’s solutions.

*“During the exam” is defined as: After you start the exam, and before you turn in the exam and leave the testing site.

- You have 90 minutes to complete this exam. If the course staff must ask you to stop writing or to turn in your exam more than once after time is called, you may receive a score of zero.
- After you have turned in your exam, you may not discuss the contents of this exam with ANYONE other than the course staff until 9:00 PM tonight.
- If you need to make a phone call during the exam, ask a proctor.

I understand and accept these instructions. All smart devices on my person are in airplane mode and disconnected from all wireless networks.

Signature: ____________________________________________

Remember to show your work and justify your answer if required (additional tips are on the next page). Present all solutions in as organized a manner as possible. GOOD LUCK!
Here are some tips:

- If you have time, it’s always a good idea to check your work when possible.

- If you get the wrong answer but show your work, you have a better chance of receiving partial credit.

- **DO NOT attempt to estimate any of your answers as decimals.** For example, \(1 - \frac{1}{\pi}\) is a much better answer than 0.682, because it is exact.

- The very last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

**Integral table entries you may need**

\[
\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C
\]

\[
\int \frac{du}{(u - r)(u - s)} = \frac{1}{r - s} \ln \left| \frac{u - r}{u - s} \right| + C
\]

\[
\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C
\]

\[
\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \left( \frac{u}{a} \right) + C
\]

In the entries above, \(a, r, s\) are constants such that \(a \neq 0\) and \(r \neq s\).

**Values of arcsine and arctangent**

The table below gives important values of the arcsine and arctangent functions:

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>(\frac{1}{2})</th>
<th>(\frac{1}{\sqrt{3}})</th>
<th>(\frac{1}{2})</th>
<th>(\sqrt{3}/2)</th>
<th>1</th>
<th>(\sqrt{3})</th>
<th>(\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>arcsin (x)</td>
<td>0</td>
<td>(\pi/6)</td>
<td>(\pi/4)</td>
<td>(\pi/3)</td>
<td>(\pi/2)</td>
<td>undef.</td>
<td>undef.</td>
<td></td>
</tr>
<tr>
<td>arctan (x)</td>
<td>0</td>
<td>(\pi/6)</td>
<td>(\pi/4)</td>
<td>(\pi/3)</td>
<td>(\pi/2)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For negative values: \(\arcsin(-x) = -\arcsin(x)\) and \(\arctan(-x) = -\arctan(x)\). Blank entries in the table are not “nice” multiples of \(\pi\).
Unless otherwise specified, in problems 1–7 you do not need to justify your answer.

1. For each of the following a.–d., describe all of the values of \( p \) for which the given integral converges.

a. \( \int_{0}^{1} \frac{1}{x^p} \, dx \)

\[ \text{converges for all } p < 1 \]

b. \( \int_{1}^{\infty} \frac{1}{x^p} \, dx \)

\[ \text{converges for all } p > 1 \]

c. \( \int_{0}^{\infty} \frac{1}{x^p} \, dx = \int_{0}^{1} + \int_{1}^{\infty} \) \( \leq \) must both converge.

\[ \text{converges for no values of } p \]

d. \( \int_{100}^{\infty} e^{-px} \, dx. \)

\[ \text{converges for all values of } p > 0 \]

\[ \int_{0}^{\infty} e^{-px} \, dx \text{ does, and changing the lower endpt. to } 100 \text{ does not affect convergence} \]
2. In (a–h) fill in the blank with the correct asymptotic relation: ∼, <, or >.

a. \( \frac{1}{x} \quad \overset{\underset{\curvearrowright}{\sim}}{\overset{\underset{\sim}{<}}{\sim}} \) \( \ln(x) \)

b. \( \pi^x \quad \overset{\underset{\curvearrowright}{\sim}}{\overset{\underset{\sim}{>}}{>}} \) \( 3^x \)

c. \( e^{-x} \quad \overset{\underset{\curvearrowright}{\sim}}{\overset{\underset{\sim}{>}}{>}} \) \( e^{-x^2} \)

d. \( \ln(x) \quad \overset{\underset{\sim}{<}}{\overset{\underset{\sim}{<}}{<}} \) \( x^{1/1000} \)

e. \( x + \frac{1}{x} \quad \overset{\underset{\sim}{<}}{\overset{\underset{\sim}{<}}{<}} \) \( x \)

f. \( \ln(x^5) \quad \overset{\underset{\sim}{<}}{\overset{\underset{\sim}{<}}{<}} \) \( \ln(x^{1/5}) \)

g. \( x \ln(x) \quad \overset{\underset{\sim}{>}}{\overset{\underset{\sim}{>}}{>}} \) \( 100^{100^{100}} x \)

h. \( \sqrt{x^2 + 1} \quad \overset{\underset{\sim}{<}}{\overset{\underset{\sim}{<}}{<}} \) \( x^{0.001} \)

3. Suppose that \( f(x) \) and \( g(x) \) are “nice” (defined, continuous, and positive) functions on \([0, \infty)\). Suppose furthermore that

\[ f(x) < g(x) \quad \text{as} \quad x \to \infty. \]

Which of the following statements must be true, given the information above?

Circle all true statements.

i. If \( \int_0^\infty f(x) \, dx \) and \( \int_0^\infty g(x) \, dx \) both converge, then \( \int_0^\infty f(x) \, dx \leq \int_0^\infty g(x) \, dx \).

ii. If \( \int_0^\infty f(x) \, dx \) converges, then \( \int_0^\infty g(x) \, dx \) also converges.

iii. If \( \int_0^\infty g(x) \, dx \) converges, then \( \int_0^\infty f(x) \, dx \) also converges.

\( \leftarrow \text{limit comp} \)

iv. If \( \lim_{x \to \infty} \left( \frac{f(x)}{g(x)} \right) = 0 \) then \( \int_0^\infty \frac{f(x)}{g(x)} \, dx \) converges.

v. If \( \int_0^\infty \frac{1}{f(x)} \, dx \) converges, then \( \int_0^\infty \frac{1}{g(x)} \, dx \) also converges.

\( \leftarrow \frac{1}{f} > \frac{1}{g} \), then \( \lim \text{comp} \)

vi. If \( \int_0^\infty f(x) \, dx \) converges, then \( \int_0^\infty f(x)^2 \, dx \) also converges.

vii. None of the above.
4. a. In a single picture, sketch the graphs $y = \sin(x)$ and $y = x$ for $x$ between $-\pi/2$ and $\pi/2$.

b. For $x$ between 0 and $\pi/2$, we have the inequalities

$$0 \leq \sin(x) \leq x.$$

Explain using your answer in (a) why the inequality above holds.

Hint: if you can’t see this inequality in your picture from part a., you should redraw it.

\[ y = x \text{ is tangent to } y = \sin x \text{ at } x = 0, \text{ and } \sin x \text{ is concave down on the interval } 0 \leq x \leq \frac{\pi}{2}. \]

c. Does $\int_0^{\pi/2} \frac{1}{\sin x} \, dx$ converge or diverge? Justify your answer.

Hint: you should use the inequality from part b.

It diverges, b/c $0 \leq \sin x \leq x$

implies $\frac{1}{x} \leq \frac{1}{\sin x}$ on the interval of integration.

So $\int_0^{\pi/2} \frac{1}{\sin x} \, dx$ diverges, since $\int_0^{\pi/2} \frac{1}{x} \, dx$ does also.
5. What is the value of the geometric sum

\[ 3^7 - 3^8 + 3^9 - 3^{10} + \ldots + 3^{121} ? \]

Once you’ve applied the geometric sum formula you do not need to simplify your answer. Draw a box around your answer.

\[
\text{geometric sum} = \sum_{k=0}^{114} (-3)^k
\]

\[
= 3^7 \left( 1 - 3 + 3^2 - 3^3 + \ldots + 3^{115} \right)
\]

\[
= 3^7 \left( \frac{1 - (-3)^{115}}{1 - (-3)} \right)
\]

6. Circle the geometric series below that diverge:

- a. \( 1 - 1 + 1 - 1 + 1 - 1 + \ldots \) \( r = -1 \)
- b. \( 1 - 0.9 + 0.81 - 0.729 + \ldots \) \( r = 0.9 \)
- c. \( 1000 + 100 + 10 + 1 + 1/10 + \ldots \) \( r = \frac{1}{10} \)
- d. \( \sum_{k=0}^{\infty} \frac{1}{17} \cdot 3^n \) \( r = 3 \)
- e. \( \sum_{k=0}^{\infty} 10^{20} \cdot \frac{1}{2^n} \) \( r = \frac{1}{2} \)

7. We can write the number \( 0.\overline{15} = 0.151515 \ldots \) as an infinite geometric series

\[ 0.\overline{15} = \frac{15}{100} + \frac{15}{100^2} + \frac{15}{100^3} + \ldots \]

a. What are the values of \( a \) and \( r \) such that this series is equal to \( \sum_{k=0}^{\infty} ar^k \)?

\[ a = \frac{15}{100} \quad r = \frac{1}{100} \]
b. Compute the value of this geometric series in order to express $0.\overline{15}$ as a fraction. Simplify the fraction to lowest terms, and draw a box around your final answer.

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad (\text{if } |r| < 1)$$ so, plugging in

from (a), we have

$$\frac{15}{99/100} = \frac{15}{99} = \frac{5}{33}$$

In Problems 8–10, evaluate the improper integral. Show your work. If you believe the integral to diverge, write DIVERGES as your final answer. Draw boxes around your final answers.

8. $\int_{-1}^{2} \frac{1}{x^2} \, dx = \int_{-1}^{0} \frac{1}{x^2} \, dx + \int_{0}^{2} \frac{1}{x^2} \, dx$

$\mathbf{\text{DIVERGES}}$

9. $\int_{0}^{\infty} \frac{e^x}{1+e^{2x}} \, dx$. Hint: $e^{2x} = (e^x)^2$.

$$\int \frac{e^x}{1+(e^x)^2} \, dx = \int \frac{du}{1+u^2} = \arctan(u) + C$$

$u = e^x$

$du = e^x \, dx$

so

$$\int_{0}^{\infty} \frac{e^x}{1+e^{2x}} \, dx = \lim_{b \to \infty} \left[ \arctan(e^b) - \arctan(e^0) \right]$$

$$= \arctan(\infty) - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$
10. \[ \int_4^\infty \frac{dx}{x^2 - x - 6} \quad \text{Hint:} \quad \frac{5}{x^2 - x - 6} = \frac{1}{x - 3} - \frac{1}{x + 2} \]

\[ \int_4^\infty \frac{dx}{x^2 - x - 6} = \frac{1}{5} \int_4^\infty \left( \frac{1}{x-3} - \frac{1}{x+2} \right) \, dx \]

\[ = \frac{1}{5} \lim_{b \to \infty} \left[ \ln |x-3| - \ln |x+2| \right]_4^b \]

\[ = \frac{1}{5} \lim_{b \to \infty} \left[ \ln \left| \frac{x-3}{x+2} \right| \right]_4^b \]

\[ = \frac{1}{5} \left[ \ln \left( \lim_{b \to \infty} \frac{b-3}{b+2} \right) - \ln \left| \frac{4-3}{4+2} \right| \right] \]

\[ = \frac{1}{5} \left[ \ln (1) - \ln \left( \frac{1}{6} \right) \right] \]

\[ = \frac{1}{5} \left[ 0 - \ln \left( \frac{1}{6} \right) \right] \]

\[ = \frac{1}{5} \ln (6) \]
In Problems 11–13, determine whether the integral **CONVERGES** or **DIVERGES** using the method of your choice. Justify your answer!

- If you evaluate the integral, show all steps.
- If you use limit or direct comparison, outline your argument well enough that we can understand how you came to your conclusion.
- You may use the convergence conditions for \( \int_0^1 \frac{1}{x^p} \, dx \), \( \int_1^\infty \frac{1}{x^p} \, dx \), \( \int_0^\infty r^{-x} \, dx \), and \( \int_0^\infty e^{-ax} \, dx \) without justification, but mention those conditions if you use them.

**Draw boxes around your final answers.**

11. \[ \int_1^\infty \frac{x^{17} + e^{x^2} + 9}{e^{3x} + \ln(x)} \, dx \]

   dominant terms

   \[ \sqrt{\ldots} \times \sqrt{\frac{e^x}{e^{3x}}} = \frac{1}{e^x} \to \infty \]

   \( \int_1^\infty \frac{1}{e^x} \, dx = \int_1^\infty e^{-x} \, dx \) **converges**.

   by limit comparison

12. \[ \int_0^{10} \frac{t^2 + 1 + \cos t}{t^3} \, dt \]

   \[ 0 \leq 1 + \cos t \leq 2 \], so

   \[ \frac{t^2}{t^3} = \frac{t^2 + 1 + \cos t}{t^3} \leq \frac{t^2 + 2}{t^3} \]. Integral **diverges**

   because \( \int_0^{10} \frac{1}{t} \, dt \) diverges

   \( (p \text{-test} \text{, } w/ \text{ } p = 1) \).
13. \[ \int_0^\infty \frac{dx}{5x^2 + x + x^{1/3}} \]

Have to split, b/c two issues:

- Asympt. at \( x = 0 \)
- Integral has \( \infty \) at the top.

\[ A = \int_0^1 \frac{dx}{5x^2 + x + x^{1/3}} + \int_1^\infty \frac{dx}{5x^2 + x + x^{1/3}} \]

\( A \) converges by \( \lim \) comp. w/ \( \int_1^\infty \frac{1}{x^2} \, dx \)

\[ B = \int_0^1 \frac{1}{x^{1/3}} \, dx \]

Problem: divergent.

\[ \frac{1}{7} \leq \frac{1}{5x^{5/3} + x^{2/3} + 1} \leq \frac{1}{x^{1/3}} \quad \text{when} \quad 0 \leq x \leq 1 \]

(at \( x = 1 \))

\[ \frac{1}{7x^{1/3}} \leq \text{integrand} \leq \frac{1}{x^{1/3}} \]

\( B \) converges, \( A + B \) converges.

Since both \( A \) and \( B \) converge, \( A + B \) converges too.