Failure to follow the instructions below will constitute a breach of the Stanford Honor Code:

• DO NOT WRITE ANYTHING ON THIS PAGE OTHER THAN YOUR NAME, SEAT NUMBER, SIGNATURE, AND STUDENT ID.
  • You may not use a calculator or any notes or book during the exam.
  • You may not access your cell phone or any other electronics during the exam for any reason.
  • You must sit in your assigned seat.
  • You may not communicate with anyone other than the course staff during the exam, or look at anyone else’s solutions.

• DO NOT DETACH ANY PAGES OF THE EXAM.

I understand and accept these instructions.

Signature: _______________________________________________________

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Tips:

• Show your work and justify your answer if required. Present all solutions in as organized a manner as possible.

• DO NOT attempt to estimate any of your answers as decimals. For example, \( \frac{\pi^2}{12} \) is a much better answer than 0.822, because it is exact.

• Pages 2 and 10 of the exam are blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

DO NOT DETACH ANY PAGES OF THE EXAM.

GOOD LUCK!
In Problems 1–7 you do not need to justify or explain your answer unless otherwise specified.

1. a. Write down an example of an infinite series that is conditionally (and not absolutely) convergent.

   b. Which number \( r \) satisfies \( 1 + r + r^2 + r^3 + \cdots = 6 \)? Draw a box around your answer.

2. Let \( s \) be a positive real number.

   a. For which values of \( s \) does the series \( \sum_{n=1}^{\infty} \frac{1}{n^s} \) converge? Describe all such values.

   b. For which values of \( s \) does the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \) converge? Describe all such values.
3. In order to apply the integral test to the series \( \sum_{n=1}^{\infty} f(n) \) for convergence one must first verify that \( f(x) \)
has what property/properties? List them.

4. The infinite series
\[
(\star) \sum_{n=1}^{\infty} \frac{1}{2n^2} \quad \text{and} \quad (\star\star) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}
\]
both converge to \( \frac{\pi^2}{12} \).

The integral test gives the following bounds on the error for the \( N \)th partial sum of (\( \star \)):
\[
\int_{N+1}^{\infty} \frac{dx}{2x^2} \leq \frac{\pi^2}{12} - \sum_{n=1}^{N} \frac{1}{2n^2} \leq \frac{1}{2(N+1)^2} + \int_{N+1}^{\infty} \frac{dx}{2x^2}
\]

The alternating series test gives the following bounds on the error for the \( N \)th partial sum of (\( \star\star \)):
\[
\left| \frac{\pi^2}{12} - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2} \right| \leq \frac{1}{(N+1)^2}
\]

Which of the following is a better estimate for \( \frac{\pi^2}{12} \)? Circle the correct answer.

i. The 999th partial sum of series (\( \star \)).

ii. The 999th partial sum of series (\( \star\star \)).

iii. The 999th partial sums of (\( \star \)) and (\( \star\star \)) give equally good estimates since the infinite series converge to the same value.

iv. There is not enough information to compare the partial sums of these series without a calculator (and calculators are not allowed on the exam).

You do not need to explain your answer.
5. Suppose that the sequence \( \{a_n\} \) consists of positive terms and is decreasing.

Suppose furthermore that the infinite series \( \sum_{n=0}^{\infty} a_n \) converges.

Which of the following must be true, given the information provided? Circle all true statements.

i. \( \lim_{n \to \infty} (a_n) = 0 \).

ii. The series \( \sum_{n=0}^{\infty} (a_n + 1) \) converges.

iii. The series \( \sum_{n=0}^{\infty} (-1)^n a_n \) converges.

iv. The series \( \sum_{n=0}^{\infty} |a_n| \) converges.

v. \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \).

vi. The series \( \sum_{n=0}^{\infty} a_n^2 \) converges.

6. Which of the following power series represents \( \frac{4}{1 + x^2} \)? Circle the correct answer.

i. \( \sum_{n=0}^{\infty} (-1)^n x^{4n} = 1 - x^4 + x^8 - x^{12} + \ldots \)

ii. \( \sum_{n=0}^{\infty} 4x^{n+2} = 4x^2 + 4x^3 + 4x^4 + 4x^5 + \ldots \)

iii. \( \sum_{n=0}^{\infty} 4(-1)^n x^{n+2} = 4x^2 - 4x^3 + 4x^4 - 4x^5 + \ldots \)

iv. \( \sum_{n=0}^{\infty} 4(-1)^n x^{2n} = 4 - 4x^2 + 4x^4 - 4x^6 + \ldots \)

v. \( \sum_{n=0}^{\infty} (-1)^n (4x^2)^n = 1 - 4x^2 + 16x^4 - 64x^6 + \ldots \)

vi. \( \sum_{n=0}^{\infty} (4x^2)^n = 1 + 4x^2 + 16x^4 + 64x^6 + \ldots \)

7. To what value does the infinite series \( \sum_{n=0}^{\infty} \frac{1}{2^n \cdot n!} \) converge? Write it below.
For 8–14 state whether the given series converges absolutely, converges conditionally, or diverges. State which test you used and show the work necessary to justify your answer.

8. \( \sum_{n=2}^{\infty} \frac{(-1)^n \cdot n}{\sqrt{n}} \)

9. \( \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n + 1} \right) \)

10. \( \sum_{n=1}^{\infty} \frac{(n + 1)^2}{n^3} \)
11. \( \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^3 + 1}} \)

12. \( \sum_{n=1}^{\infty} \frac{4^n}{n + 3^n} \)
13. $\sum_{n=1}^{\infty} \frac{1 + (\cos n)^2}{n}$

14. $\sum_{n=0}^{\infty} \frac{(2n)!}{5^n (n!)^2}$
15. Determine the interval of convergence of the power series \[ \sum_{n=1}^{\infty} \frac{(-1)^n (x - 1)^n}{5^n \cdot n^2}. \]

*Show your work and draw a box around your final answer.*