Math 21, Winter 2017 — Schaeffer
Practice Midterm Exam 2 (February 22nd, 2017)

Last/Family Name | First/Given Name
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Failure to follow the instructions below will constitute a breach of the Stanford Honor Code:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone or any other electronics during the exam for any reason.
- You must sit in your assigned seat.
- You may not communicate with anyone other than the course staff during the exam, or look at anyone else’s solutions.
- You have 80 minutes to complete this exam.

I will not discuss this exam with ANYONE other than the course staff until: ____________________
I understand and accept these instructions.

Signature: _______________________________________________________

Remember to show your work and justify your answer if required (additional tips are on the next page). Present all solutions in as organized a manner as possible.

GOOD LUCK!
Here are some tips:

• If you have time, it’s always a good idea to check your work.

• If you get the wrong answer for an integral but show your work, chances are good that we can award you partial credit.

• DO NOT attempt to estimate any of your answers as decimals. For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is exact.

• The boxes at the end of each topic are for grading purposes only. Do not touch or look at these boxes. Pretend they are not there.

• The last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

Bounds from the integral and alternating series tests

• Provided that $f(x)$ satisfies the conditions required to apply the alternating series test (on $[N, \infty)$), if $\sum_{n=N}^{\infty} f(n)$ converges, then

$$\int_{N}^{\infty} f(x) \, dx \leq \sum_{n=N}^{\infty} f(n) \leq f(N) + \int_{N}^{\infty} f(x) \, dx$$

• If $\sum a_n$ is an alternating series and it converges, then

$$|S - S_N| \leq |a_{N+1}|$$

where $S = \sum_{n=K}^{\infty} a_n$ and $S_N = \sum_{n=K}^{N} a_n$. 
In Problems 1–7 you do not need to justify or explain your answer unless otherwise specified.

1. Let \( p > 0 \). Which of the following statements is true about the infinite series
   \[
   \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}
   \]
   i. It diverges if \( p > 1 \), converges conditionally if \( p = 1 \), and converges absolutely if \( 0 < p < 1 \).
   ii. It diverges if \( 0 < p < 1 \) and converges absolutely if \( p \geq 1 \).
   iii. It diverges if \( 0 < p \leq 1 \) and converges absolutely if \( p > 1 \).
   iv. It converges conditionally if \( 0 < p \leq 1 \) and converges absolutely if \( p > 1 \).
   v. It converges diverges for \( p = 1 \) and converges for all other values of \( p > 0 \).

2. Suppose you know that \( 2^n + 2 \leq 3^n \) for \( n \geq 2 \). Which of the following conclusions is correct?
   i. \( \sum_{n=2}^{\infty} \frac{1}{2^n + 2} \) diverges.
   ii. \( \sum_{n=2}^{\infty} \frac{1}{2^n + 2} \) converges, and to a value \( \leq \sum_{n=2}^{\infty} \frac{1}{3^n} = \frac{1}{6} \).
   iii. \( \sum_{n=2}^{\infty} \frac{1}{2^n + 2} \) converges, and to a value \( \geq \sum_{n=2}^{\infty} \frac{1}{3^n} = \frac{1}{6} \).
   iv. None of the above.

3. Partario and Quintana are trying to determine whether the series
   \[
   \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right)
   \]
   converges or diverges. Here are their solutions:

   **PARTARIO’S SOLUTION:** Begin by splitting the sum up:
   \[
   \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{k=1}^{\infty} \frac{1}{k+1}
   \]
   both \( \sum \frac{1}{k} \) and \( \sum \frac{1}{k+1} \) diverge by the Integral Test, so the original series diverges too.

   **QUINTANA’S SOLUTION:** We have
   \[
   \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \cdots + \left( \frac{1}{n-1} - \frac{1}{n} \right) + \left( \frac{1}{n} - \frac{1}{n+1} \right)
   \]
   Canceling the second term of each (\( \cdots \)) with the first term of the next (\( \cdots \)) gives us the formula
   \[
   \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}
   \]
   Since \( \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1 \), the series diverges by the Divergence Test.
a. Does the series converge or diverge? Which test did you use?

b. If you found that the series converges, explain why Partario’s solution is incorrect.

c. If you found that the series converges, explain why Quintana’s solution is incorrect.

4. Consider the series \( \sum_{n=0}^{\infty} \frac{1}{(3n - 4)^{3}} \).

a. What is the least \( N \) so that the integral test applies to the series \( \sum_{n=N}^{\infty} \frac{1}{(3n - 4)^{3}} \)?

(You do not need to justify your answer.)

b. Use part (a.) and the bounds for the integral test (on the 2nd page) to find bounds on \( \sum_{n=0}^{\infty} \frac{1}{(3n - 4)^{3}} \).

5. In HW5 we learned the power series representation

\[
\ln(x + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{n}}{n} = x - \frac{x^{2}}{2} + \frac{x^{3}}{3} - \frac{x^{4}}{4} + \cdots
\]

valid on the series’ interval of convergence. Using this information, which of the following series converges to \( \ln(3) \)? (There may be more than one.)

i. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^{n}}{n} \)

ii. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^{n}}{n} \)

iii. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^{n}}{n \cdot 3^{n}} \)

iv. \( \sum_{n=1}^{\infty} \frac{2^{n}}{n \cdot 3^{n}} \)

v. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot 3^{n}} \)

vi. None of the above.

6. Consider the power series \( \sum_{n=0}^{\infty} (-1)^{n} (4x)^{2n+1} \)

a. What is this power series’ radius of convergence equal to?

i. \( \frac{1}{2} \)

ii. 1

iii. 2

iv. 4

v. None of the above.

b. Which function does the power series represent on its interval of convergence?

i. \( \frac{1}{1+4x} \)
ii. \( \frac{1}{1-4x} \)

iii. \( \frac{4x}{1+4x} \)

iv. \( \frac{4x}{1-4x} \)

v. \( \frac{4x}{1+(4x)^2} \)

vi. \( \frac{4x}{1-(4x)^2} \)

7. To what value does the series \( \sum_{n=0}^{\infty} \frac{3^n}{n!} \) converge?

For 8–14 state whether the given series converges absolutely, converges conditionally, or diverges. State which test you used and show the work necessary to prove justify your answer.

8. \( \sum_{n=0}^{\infty} \frac{2^n + (-1)^n}{n^2 + 2^n} \)

9. \( \sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{\sqrt{3n+1}} \)

10. \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) \( \text{Hint:} \frac{1}{x \ln x} \text{ is decreasing for all } x > 1. \)

11. \( \sum_{n=1}^{\infty} \frac{n!}{(3^n)^2} \)

12. \( \sum_{n=2}^{\infty} \frac{(n-1)^2}{(n^2 + 1)^2} \)

13. \( \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n}{2^n \cdot (-3)^n} \)

14. \( \sum_{n=0}^{\infty} \frac{n! \cdot (2n)!}{(3n)!} \)

15. On what interval does the power series

\[ F(x) = \sum_{n=0}^{\infty} \frac{4^n(x-1)^n}{2n+1} \]

converge? If the power series converges at one or both endpoints, specify whether convergence is absolute or conditional. Justify your answer.