In Problems 1–7 you do not need to justify or explain your answer unless otherwise specified.

1. Let \( p > 0 \). Which of the following statements is true about the infinite series

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}
\]

i. It diverges if \( p > 1 \), converges conditionally if \( p = 1 \), and converges absolutely if \( 0 < p < 1 \).
ii. It diverges if \( 0 < p < 1 \) and converges absolutely if \( p \geq 1 \).
iii. It diverges if \( 0 < p \leq 1 \) and converges absolutely if \( p > 1 \).
iv. It converges conditionally if \( 0 < p \leq 1 \) and converges absolutely if \( p > 1 \).
v. It converges diverges for \( p = 1 \) and converges for all other values of \( p > 0 \).

2. Suppose you know that \( 2^n + 2 \leq 3^n \) for \( n \geq 2 \). Which of the following conclusions is correct?

i. \( \sum_{n=2}^{\infty} \frac{1}{2^n + 2} \) diverges.

ii. \( \sum_{n=2}^{\infty} \frac{1}{2^n + 2} \) converges, and to a value \( \leq \sum_{n=2}^{\infty} \frac{1}{3^n} = \frac{1}{6} \)

\[
\text{Since } 2^n + 2 \leq 3^n \text{ for } n \geq 2,
\]

iii. \( \sum_{n=2}^{\infty} \frac{1}{2^n + 2} \) converges, and to a value \( \geq \sum_{n=2}^{\infty} \frac{1}{3^n} = \frac{1}{6} \)

iv. None of the above.

3. Partario and Quintana are trying to determine whether the series

\[
\sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right)
\]

converges or diverges. Here are their solutions:

**PARTARIO'S SOLUTION:** Begin by splitting the sum up:

\[
\sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{k=1}^{\infty} \frac{1}{k+1}
\]

both \( \sum \frac{1}{k} \) and \( \sum \frac{1}{k+1} \) diverge by the Integral Test, so the original series diverges too.

**QUINTANA'S SOLUTION:** We have

\[
\sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)
\]

Canceling the second term of each (\( \cdots \)) with the first term of the next (\( \cdots \)) gives us the formula

\[
\sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}
\]

Since \( \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1 \), the series diverges by the Divergence Test.
a. Does the series converge or diverge? Which test did you use?

b. If you found that the series converges, explain why Partario's solution is incorrect.

c. If you found that the series converges, explain why Quintana's solution is incorrect.

4. Consider the series \( \sum_{n=0}^{\infty} \frac{1}{(3n-4)^3} \).

a. What is the least \( N \) so that the integral test applies to the series \( \sum_{n=N}^{\infty} \frac{1}{(3n-4)^3} \)?

(You do not need to justify your answer.)

b. Use part (a.) and the bounds for the integral test (on the 2nd page) to find bounds on \( \sum_{n=0}^{\infty} \frac{1}{(3n-4)^3} \).

5. In HW5 we learned the power series representation

\[
\ln(x + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots
\]

valid on the series' interval of convergence. Using this information, which of the following series converges to \( \ln(3) \)? (There may be more than one.)

i. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^n}{n} \) ← diverges! (by D.T. or R.T.)

ii. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n} \)

iii. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n \cdot 3^n} = \ln \left( \frac{2}{3} + 1 \right) = \ln \left( \frac{5}{3} \right) \)

iv. \( \sum_{n=1}^{\infty} \frac{2^n}{n \cdot 3^n} = -\ln \left( -\frac{2}{3} + 1 \right) = -\ln \left( \frac{1}{3} \right) = \ln (3) \)

v. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 1^n}{n \cdot 3^n} = \ln \left( \frac{1}{3} + 1 \right) = \ln \left( \frac{4}{3} \right) \)

vi. None of the above.

6. Consider the power series \( \sum_{n=0}^{\infty} (-1)^n (4x)^{2n+1} \).

a. What is this power series' radius of convergence equal to?

i. \( \frac{1}{2} \)

ii. \( 1 \)

iii. \( 2 \)

iv. \( 4 \)

(\( R = \frac{1}{4} \))

b. Which function does the power series represent on its interval of convergence?

i. \( \frac{1}{1+4x} = \sum_{n=0}^{\infty} (-1)^n (4x)^n \)
ii. \[ \frac{4x}{1+4x} = \sum_{n=0}^{\infty} (-1)^n (4x)^{n+1} \]

iii. \[ \frac{4x}{1+(4x)^2} = \sum_{n=0}^{\infty} (4x)^n \]

iv. \[ \frac{4x}{1-(4x)^2} = \sum_{n=0}^{\infty} (4x)^{2n+1} \]

vi. \[ \sum_{n=0}^{\infty} (2^n x^n) \]

7. To what value does the series \[ \sum_{n=0}^{\infty} \frac{2^n}{n!} \] converge? = \[ e^3 \] (plugging \( x = 3 \) into the identity \( \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \))

For 8–14 state whether the given series converges absolutely, converges conditionally, or diverges. State which test you used and show the work necessary to prove justify your answer.

8. \[ \sum_{n=0}^{\infty} \frac{2^n + (-1)^n}{n^2 + 2^n} \]

9. \[ \sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{\sqrt{3n + 1}} \]

10. \[ \sum_{n=2}^{\infty} \frac{1}{n \ln n} \] Hint: \( \frac{1}{x \ln x} \) is decreasing for all \( x > 1 \). See Eq. 5

11. \[ \sum_{n=1}^{\infty} \frac{n!}{(3n)^2} \]

12. \[ \sum_{n=2}^{\infty} \frac{(n-1)^2}{(n^2 + 1)^2} \]

13. \[ \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n}{2^n \cdot (-3)^n} \]

14. \[ \sum_{n=0}^{\infty} \frac{n! \cdot (2n)!}{(3n)!} \]

15. On what interval does the power series \[ F(x) = \sum_{n=0}^{\infty} \frac{4^n(x-1)^n}{2n+1} \] converge? If the power series converges at one or both endpoints, specify whether convergence is absolute or conditional. Justify your answer.

See Eq. 5
The series converges:
\[
\sum_{k=0}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=0}^{\infty} \frac{(k+1)-k}{k(k+1)} = \sum_{k=0}^{\infty} \frac{1}{k(k+1)}
\]
Since \( \frac{1}{k(k+1)} \leq \frac{1}{k^2} \) as \( k \to \infty \) and \( \sum_{k=0}^{\infty} \frac{1}{k^2} \) converges, the series converges.

Part (b): Quintana is incorrect b/c you can’t split
\[
\sum_{k=0}^{\infty} (a_n + b_n) = \sum_{k=0}^{\infty} a_n + \sum_{k=0}^{\infty} b_n
\]
unless both \( \sum a_n \) and \( \sum b_n \) converge. Since neither does, this is an illegal move.

Quintana’s formula
\[
\sum_{k=0}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}
\]
is correct; however, in her implementation of the D.T. she takes
\[
\lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right), \text{ but was supposed to take the limit of the terms, i.e.}\]
\[
\lim_{k \to \infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 0.
\]
In fact, Quintana’s argument, with some fixin’, shows
\[
\sum_{k=0}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1
\]
\[
\text{limit of partial sums}
\]

Writing out terms,
\[
\sum_{n=0}^{\infty} \frac{1}{(3n-4)^3} = \frac{1}{(-4)^3} + \frac{1}{(-1)^3} + \frac{1}{2^3} + \frac{1}{5^3} + \ldots
\]
\[
= -\frac{1}{64} - 1 + \left[ \frac{1}{8} + \frac{1}{125} + \ldots \right]
\]
leads us to suspect \( n=2 \). Indeed:
If \( n \geq 2 \), then \( 3n-4 \geq 2 \) so \( (3n-4)^3 \geq 8 \)
so \( \frac{1}{(3n-4)^3} \) is positive. \( (3x-4)^3 \) is increasing for all \( x \), so \( (3x-4)^3 \) is, and \( \frac{1}{(3x-4)^3} \) is decreasing.
Finally, \( \frac{1}{(3x-4)^3} \) is etc. except at \( x = \frac{4}{3} \).
4) Thus \( N = 2 \)

(b) We have

\[
\sum_{n=0}^{\infty} \frac{1}{(3n-4)^3} = -\frac{1}{64} - 1 + \sum_{n=2}^{\infty} \frac{1}{(3n-4)^3}
\]

Since \( \frac{1}{(3x-4)^3} \) is pos. decr. cts. on \([2, \infty)\),

\[
\int_{2}^{\infty} \frac{dx}{(3x-4)^3} \leq \sum_{n=2}^{\infty} \frac{1}{(3n-4)^3} \leq \frac{1}{8} + \int_{2}^{\infty} \frac{dx}{(3x-4)^3}
\]

\[
\int u = 3x - 4, \quad du = 3 \, dx
\]

\[
\frac{1}{24} \leq \sum_{n=2}^{\infty} \leq \frac{1}{8} + \frac{1}{24}
\]

Thus,

\[-\frac{1}{64} - 1 + \frac{1}{24} \leq \sum_{n=0}^{\infty} \frac{1}{(3n-4)^3} \leq -\frac{1}{64} - 1 + \frac{1}{8} + \frac{1}{24}
\]

8) **Diverges by D.T.**

\[
\lim_{n \to \infty} \left( \frac{2^n + (-1)^n}{n^2 + 2^n} \right) = \lim_{n \to \infty} \left( \frac{1 + \frac{(-1)^n}{2^n}}{\frac{n^2}{2^n} + 1} \right) = \frac{1 + 0}{0 + 1} = 1 \neq 0.
\]

9) **Converges conditionally.**

\[
\sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{\sqrt{3n+1}}
\]

\((3n+1 = 4, 7, 10, \ldots \text{ for } n = 1, 2, 3, \ldots)\)

\((\text{odd, odd, odd...})\)

\[
\left| \frac{(-1)^{3n+1}}{\sqrt{3n+1}} \right| = \frac{1}{\sqrt{3n+1}} \text{ is } A \text{ with limit } 0. \text{ Thus, the series converges by AST}
\]

Convergence is **conditional**, b/c

\[
\sum_{n=1}^{\infty} \left| \frac{(-1)^{3n+1}}{\sqrt{3n+1}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{3n+1}}
\]

diverges by LCT \( \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \)

10) **Hint and presence of \( \ln(n) \) suggest Integral Test!**

\[
\frac{1}{\ln x} \text{ is pos., decr., and cts. on } [2, \infty)
\]

\[
\text{as } x \to 0^{+}, \quad \ln x \to 0 \text{ by \( x = 1 \)}
\]

\[
\text{asymptotes at } x = 0 \text{ and } x = 1
\]

so: Consider

\[
\int_{2}^{\infty} \frac{dx}{\ln x} = \lim_{b \to \infty} \left[ \ln \ln x \right]_{2}^{b}
\]

Thus, since the

\[
\int_{2}^{\infty} \frac{dx}{\ln x} = \lim_{b \to \infty} \left[ \ln \ln x \right]_{2}^{b}
\]

\(\infty\)

\(\int \text{ diverges, } \sum \text{ diverges, too.}\)
11. Ratio Test:

\[ L = \lim_{n \to \infty} \left| \frac{(n+1)!}{(3n+3)!} \cdot \frac{(3n)!}{n! (2n)!} \right| \]

\[ = \lim_{n \to \infty} \left| \frac{n+1}{3^2} \right| = \frac{1}{9} \lim_{n \to \infty} \left| \frac{n+1}{n+1} \right| = \frac{1}{9} \]

So, \( \sum \) diverges.

12. Converges:

\[ \sum_{n=1}^{\infty} \frac{(n-1)^2}{(n^2+1)^2} = \sum_{n=2}^{\infty} \frac{n^2-2n+1}{n^4+2n^2+1} \]

Since \( \frac{n^2+\ldots}{n^4+\ldots} < \frac{1}{n^2} \), and \( \sum \frac{1}{n^2} \) converges, this series converges too.

13. Simplifying:

\[ \frac{(-1)^n 4^n}{2^n (-3)^n} = \frac{4^n}{2^n \cdot 3^n} = \frac{4^n}{6^n} = \left( \frac{2}{3} \right)^n \]

So, series is \( \sum \left( \frac{2}{3} \right)^n \) which is geometric with \( r = \frac{2}{3} \), so converges.

14. Ratio Test:

\[ L = \lim_{n \to \infty} \left| \frac{(n+1)! (2n+2)! (3n)!}{(3n+3)!} \cdot \frac{n! (2n)!}{n! (2n)!} \right| \]

\[ = \lim_{n \to \infty} \left| \frac{(n+1)(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)} \right| \]

\[ = \lim_{n \to \infty} \left| \frac{4n^3+\ldots}{27n^3+\ldots} \right| = \frac{4}{27} \]

Since \( L < 1 \), the series converges.

15. Center: at \( [1, \infty) \)

Answer: \( [\frac{3}{4}, \frac{5}{4}] \)

Ratio Test:

\[ L = \lim_{n \to \infty} \left| \frac{4^{n+1} (x-1)^{n+1}}{2n+3} \cdot \frac{2n+1}{4^n (x-1)^n} \right| \]

\[ = \lim_{n \to \infty} \left| \frac{4(x-1)(2n+1)}{2n+3} \right| = 4 |x-1| \cdot \lim_{n \to \infty} \left| \frac{2n+3}{2n+1} \right| \]

\[ = 4 |x-1|, \]

So, converges absolutely if \( 4|x-1| < 1 \)

\( |x-1| < \frac{1}{4} \) Radius: \( \frac{1}{4} \)

Endpoints: at \( x = 1 + \frac{1}{4} \), series is

\[ \sum_{n=0}^{\infty} \frac{4^n (x+1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \]

which converges (AST). At \( x = 1 + \frac{1}{4} \), \( \sum \frac{1}{2n+1} \), which diverges (LCT).