1. Consider the alternating $p$-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$$

a. For which values of $p$ does the series above converge absolutely?

$$p > 1$$

b. For which values of $p$ does the series above converge conditionally?

$$0 < p \leq 1$$

2. a. Complete the statement of the alternating series test by filling in the two blanks:

*If the sequence \( \{a_k\} = \{a_0, a_1, a_2, \ldots\} \) is*

i. positive,

ii. decreasing, and

iii. \( \lim_{k \to \infty} (a_k) = 0 \)

*then the series \( \sum_{k=0}^{\infty} (-1)^k a_k \) converges.*
b. If the series $S = \sum_{k=0}^{\infty} (-1)^k a_k$ as in part (a) converges by the alternating series test, then

$$|S - S_n| \leq |a_{n+1}|$$

where as usual, $S_n = \sum_{k=0}^{n} (-1)^k a_k$.

The infinite series below converges by the alternating series test:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(7n + 1)^2} = \left[ 1 - \frac{1}{512} + \frac{1}{3375} \right] + \frac{1}{10648} + \frac{1}{24389} - \frac{1}{46656} + \frac{1}{79607} - \frac{1}{125000} + \frac{1}{185193} - \cdots$$

Draw a close-bracket $|$ in the right-hand expression above so that the expression between the brackets $[ ]$ is the partial sum of the series with the least number of terms that is guaranteed (by the above error bounds) to estimate the series' value to within $10^{-4}$.

3. Let $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ be a sequence of numbers such that $-1 \leq a_n \leq 1$ for all indices $n$ (for the purpose of this problem this is all you know about the sequence).

Which of the following statements must be true? Circle all true statements.

i. $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converges absolutely.

(by D.C. w/ $\sum \frac{1}{n^2}$)

ii. $\sum_{n=1}^{\infty} \frac{|a_n| + 1}{n}$ diverges.

(by D.C. w/ $\sum \frac{1}{n}$)

iii. $\sum_{n=1}^{\infty} \frac{1}{a_n^2 + 1}$ converges (either absolutely or conditionally).

iv. $\sum_{n=1}^{\infty} \sqrt{a_n^2 + 4}$ diverges.

(all terms $\geq 2$)

v. $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges (either absolutely or conditionally).

vi. $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$ converges by direct comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

vii. $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$ diverges by direct comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

viii. None of the above statements is true.
4. Below are four equations involving infinite series, each with one unknown quantity:

\[ \sum_{n=0}^{\infty} \frac{X^n}{n!} = 3 \quad \sum_{n=0}^{\infty} \frac{(-1)^n Y^{2n}}{(2n)!} = 2 \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = Z \quad \sum_{n=0}^{\infty} W^n = -1 \]

In the spaces below, write down values for \( X \), \( Y \), \( Z \), and \( W \) that make each of the equations above true, or write DNE if no such value exists.

a. \( X = \ln(3) \)

b. \( Y = \text{DNE} \quad (\cos x = 2 \text{ has no solution}) \)

c. \( Z = \arctan(1) = \frac{\pi}{4} \)

d. \( W = \text{DNE} \quad \left( \frac{1}{1-W} = -1 \Rightarrow W = 2, \text{ but } 2 \text{ is not in the interval of convergence for our series!} \right) \)

5. Which of the following is the 3rd degree Taylor polynomial for \( \sin x \) centered at \( x = \frac{\pi}{2} \)?

Circle the correct answer.

i. \( x - \frac{x^3}{6} \)

ii. \( (x - \frac{\pi}{2}) - \frac{(x - \frac{\pi}{2})^3}{6} \)

iii. \( 1 - \frac{x^2}{2} \)

iv. \( 1 - \frac{(x - \frac{\pi}{2})^2}{2} \)

v. None of the above.
6. a. The Taylor series for \( x^5 \ln(1 + x^3) \) centered at \( x = 0 \) has the form

\[
x^a - \frac{x^b}{2} + \frac{x^c}{3} - \frac{x^d}{4} + \ldots
\]

where \( \{a, b, c, d, \ldots\} \) is an increasing sequence of positive integers—the list of the powers of \( x \) that appear in the Taylor series.

Which of the following sequences is \( \{a, b, c, d, \ldots\} \)? Circle the correct answer.

i. \( \{5n + 3\} = \{3, 8, 13, 18, \ldots\} \)

ii. \( \{n + 5\} = \{5, 6, 7, 8, \ldots\} \)

iii. \( \{15n + 5\} = \{5, 20, 35, \ldots\} \)

iv. \( \{3n + 8\} = \{8, 11, 14, 17, \ldots\} \)

v. None of the above.

Note: The explicit formulas for the sequences on the left-hand sides above all start at \( n = 0 \).

b. What is the 65th derivative of \( x^5 \ln(1 + x^3) \) at \( x = 0 \)?

Draw a box around your answer. You do not need to simplify your answer (though it may contain no unevaluated derivatives).

\[
\text{Coeff. of } x^{65} = \frac{f(65) \(0\)}{65!}
\]

so \( f(65) \(0\) = 65! \cdot \left( \text{coeff. of } x^{65} \right) \)

since Taylor series is

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+8}}{n+1}, \text{ coeff of } 65
\]

is the coeff w/ \( n=19 \) \( (3 \cdot 19 + 8 = 65) \), so

\[
\text{answer is } \left\lfloor \frac{-65!}{20!} \right\rfloor
\]

\[
(-1)^{19} = -1.
\]
For 7–11 determine whether the given infinite series converges or diverges. Clearly state which test you used (or otherwise explain your reasoning), and show any work you believe to be necessary to justify your answer. You do not need to specify whether convergence is absolute or conditional.

7. \( \sum_{n=2}^{\infty} \frac{(-1)^n \cdot n}{\sqrt{3n+1}} \)

\[ \text{Diverges. (divergence test)} \]

\[ \lim_{n \to \infty} \left[ \frac{(-1)^n}{\sqrt{3n+1}} \right] \text{ DNE} \]

8. \( \sum_{n=2}^{\infty} \frac{1}{\sqrt[n^5 - n^4 + n^2 - 1}} \)

\[ \text{Converges. (L.C. w/p-series)} \]

Terms are \( \leq \frac{1}{n^{6/15}} \) and \( 6/5 > 1 \), so p-series \( \sum \frac{1}{n^{6/15}} \) converges.
9. \( \sum_{n=0}^{\infty} \frac{5^n \cdot (n!)^2}{(2n + 1)!} \)

\[ L = \frac{5}{4} \], which is > 1.

Diverges. Ratio test

10. \( \sum_{n=0}^{\infty} \frac{\cos n}{(2n + 1)^2} \)

Converges, since \( \sum_{n=0}^{\infty} \left| \frac{\cos n}{(2n+1)^2} \right| \) does

(by comparison w/ \( \sum \frac{1}{(2n+1)^2} \))
11. \( \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^2} \)  

Note: \( \ln n \) is positive, continuous, and increasing on \([2, \infty)\).

Terms are \( \Theta \), decreasing, and tend to zero. 
so series converges by Alt. series test.

12. Find the interval of convergence for the power series \( \sum_{n=0}^{\infty} \frac{(-1)^n(x - 3)^{2n}}{4^n(2n+1)^2} \).

Draw a box around your final answer. You do not need to specify whether the power series converges 
absolutely or conditionally at the endpoints. If you need more space, you may go on to the next page.

Ratio test yields 

\( \lim_{n \to \infty} \frac{1}{x-3} = \frac{1}{4} \) , so endpoints are 

1 and 5 (solve \( L(x) = 1 \) for \( x \))

Endpoint series are both 

\[ \sum \frac{(-1)^n (x \pm 2)^{2n}}{4^n(2n+1)^2} = \sum \frac{(-1)^n 4^n}{4^n(2n+1)^2} = \sum \frac{(-1)^n}{(2n+1)^2} \]

which converges, so 

\[ \text{IoC: } [1, 5] \text{ or } 1 \leq x \leq 5. \]