Failure to follow the instructions below is a breach of the Stanford Honor Code:

• YOU MAY NOT WRITE ANY NOTES ON THIS PAGE.
• You may not use or consult any book or notes during the exam.*
• You may not use a calculator or the calculator function on any electronic device during the exam.*
• You may not access any internet-capable electronic device during the exam,* including smartphones and smartwatches, for any reason. These devices must be switched to “airplane mode” and disconnected from all wireless networks (both cellular and wifi) during the exam*.
• You must sit in your assigned seat.
• You may not communicate with anyone other than the course staff during the exam,* or look at anyone else’s solutions.

*“During the exam” is defined as: After you start the exam, and before you turn in the exam and leave the testing site.

• You have 90 minutes to complete this exam. If course staff must ask you to turn in your exam more than twice after time is called, you will receive a score of zero.
• After you have turned in your exam, you may not discuss the contents of this exam with ANYONE other than the course staff until 9:00 PM tonight.
• If you need to make a phone call during the exam, ask a proctor.

I understand and accept these instructions. All smart devices on my person are in airplane mode and disconnected from all wireless networks.

Signature: _______________________________________________________

Remember to show your work and justify your answer if required (additional tips are on the next page). Present all solutions in as organized a manner as possible. GOOD LUCK!
Here are some tips:

- If you have time, it’s always a good idea to check your work when possible.
- If you get the wrong answer but show your work, you have a better chance of receiving partial credit.
- **DO NOT attempt to estimate any of your answers as decimals.** For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is exact.
- The very last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem’s page.

You may need the following information at some point during the exam:

- If the alternating series test gives bounds, they are of the form
  \[
  \left| \sum_{n=0}^{\infty} a_n - S_N \right| \leq |a_{N+1}|
  \]
  where $S_N = \sum_{n=0}^{N} a_n$ is the $N$th partial sum of the series.

- When the integral test gives bounds, they are of the form
  \[
  \int_a^{\infty} f(x)dx \leq \sum_{k=a}^{\infty} f(k) \leq f(a) + \int_a^{\infty} f(x)dx.
  \]

- Stirling’s approximation is $n! \approx n^{n+1/2}e^{-n}$.

**Unless otherwise specified, in problems 1–6 you do not need to justify your answer.**

1. Suppose that $s$ is a real number. Consider the infinite series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$.

In the blanks below, describe all values of $s$ for which this series converges absolutely, converges conditionally, or diverges.

<table>
<thead>
<tr>
<th>CONVERGES ABS.</th>
<th>CONVERGES COND.</th>
<th>DIVERGES</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s &gt; 1$</td>
<td>$0 &lt; s \leq 1$</td>
<td>$s \leq 0$</td>
</tr>
</tbody>
</table>
2. Let \( \{a_k\} = \{a_1, a_2, a_3, \ldots \} \) be a sequence, and let \( \{S_n\} \) be the associated sequence of partial sums:

\[
S_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n
\]

(a) Circle the true statement(s) below.

i. If \( \{a_k\} \) converges, then \( \{|a_k|\} \) must also converge.
   TRUE. The sequence \( \{a_k\} \) converges provided \( A = \lim_{k \to \infty} (a_k) \) exists and is a real number. The sequence \( \{|a_k|\} \) will converge to \( |A| \).

ii. If \( \{a_k\} \) is bounded, then \( \{a_k\} \) must converge.
   FALSE. If \( a_k = (-1)^k \) then \( \{a_k\} \) is bounded, but not convergent.

iii. If \( \{a_k\} \) is a sequence of positive numbers and \( \{S_n\} \) is bounded, then \( \{S_n\} \) must converge.
   TRUE. Because \( \{a_k\} \) consists of positive numbers, the sequence of partial sums \( \{S_n\} \) is increasing. This means \( \{S_n\} \) is monotonic. Since \( \{S_n\} \) is also bounded, it converges (since all bounded monotonic sequences converge—this is one of the two relationships between sequence properties in the study guide).

iv. If \( \{a_k\} \) converges, then \( \{S_n\} \) must also converge.
   FALSE. For example, if \( a_k = \frac{1}{k} \) then \( S_n \) diverges, because the corresponding series will be the harmonic series.

v. If \( \{S_n\} \) converges, then \( \{a_k\} \) must also converge.
   TRUE. The sequence \( \{S_n\} \) converges if and only if the series \( \sum_{k=1}^{\infty} a_k \) converges. By the divergence test, if \( \sum a_k \) converges, then \( \lim_{k \to \infty} (a_k) = 0 \). Thus, \( \{a_k\} \) converges (to zero).

vi. None of the above.

(b) Suppose we know that the sequences \( \{S_n\} \) and \( \{|S_n|\} \) both converge. Which of the following can we conclude about convergence of the infinite series? Circle the correct answer.

i. \( \sum_{k=1}^{\infty} a_k \) converges absolutely.

ii. CORRECT ANSWER. \( \sum_{k=1}^{\infty} a_k \) converges (either absolutely or conditionally).

iii. \( \sum_{k=1}^{\infty} a_k \) diverges.

iv. None of the above.

The key is to remember that \( \sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = \lim_{n \to \infty} (S_n) \). (More next page.)
That is, *the infinite series is the limit of the partial sums*. So if \( \{S_n\} \) converges, then \( \sum a_k \) converges. This narrows the field down to (i) or (ii).

*Absolute convergence* requires that \( \sum |a_k| \) converges. However! this is *not* the same as \( |\sum a_k| \). This is why (i) is incorrect.

Here’s an example: If \( a_k = (-1)^{k+1}/k \), then the corresponding series is the alternating harmonic series, which converges *conditionally* to \( \ln(2) \). That is, \( \{S_n\} \) converges to \( \ln(2) \). Therefore, \( \{|S_n|\} \) will converge to \( |\ln(2)| \). Even though both \( \{S_n\} \) and \( \{|S_n|\} \) converge, the series converges *conditionally* and does not converge *absolutely*.

3. Suppose \( f \) is a function with domain \([1, \infty)\).

(a) What property/properties must \( f \) have (at least eventually) in order to be able to use the integral test to determine whether

\[
\sum_{k=1}^{\infty} f(k)
\]

converges or diverges? Write it/them down.

\( f \) must be (eventually) positive, decreasing, and continuous.

(b) Use the bounds from the integral test to estimate the value of

\[
\sum_{k=1}^{\infty} \frac{4k}{(k^2 + 1)^3}
\]

Your answer should be an inequality like \( A \leq \sum_{k=1}^{\infty} \frac{4k}{(k^2 + 1)^3} \leq B \) for numbers \( A \) and \( B \).

You do not need to simplify your answer, but it must contain no unresolved integrals.

You do not need to formally verify the conditions from (a), which hold on all \([1, \infty)\) here.

The bounds for the integral test are on page 2. In this case, the integral is

\[
\int_1^\infty \frac{4x}{(x^2 + 1)^3} \, dx = 2 \int_1^\infty \frac{(2x \, dx)}{(x^2 + 1)^3}
\]

Now using \( u = x^2 + 1 \) and \( du = 2x \, dx \), the integral becomes

\[
= 2 \int_2^\infty \frac{du}{u^3} = 2 \cdot \lim_{b \to \infty} \left( \frac{1}{8} - \frac{1}{2b^2} \right) = \frac{1}{4}
\]

(where the lower endpoint changed because \( u = 1^2 + 1 = 2 \) when \( x = 1 \)). By the integral test bounds, the series is \( \geq 1/4 \) and \( \leq f(2) + 1/4 = 1/2 + 1/4 = 3/4 \). So:

\[
\frac{1}{4} \leq \sum_{k=1}^{\infty} \frac{4k}{(k^2 + 1)^3} \leq \frac{3}{4}
\]

which is true: The series has value 0.58139, which is between 0.25 and 0.75.

4. The integral \( \int e^{-x^2} \, dx \) (in)famously cannot be evaluated in terms of simple functions from calculus. However, we can evaluate definite integrals of \( e^{-x^2} \) in terms of infinite series. For example,

\[
\int_0^1 e^{-x^2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n + 1)}
\]

*Write down an estimate* for \( \int_0^1 e^{-x^2} \, dx \) so that the error \( \left| \int_0^1 e^{-x^2} \, dx - \text{estimate} \right| \) is guaranteed to be \( < \frac{1}{100} \). Your answer should be an arithmetic expression involving fractions of whole numbers.

*Draw a box around your answer.*

(Answer next page.)
We will use the bounds provided by the alternating series test. Expanding the series by a few terms,
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} = \frac{1}{0! \cdot 1} - \frac{1}{1! \cdot 3} + \frac{1}{2! \cdot 5} - \frac{1}{3! \cdot 7} + \frac{1}{4! \cdot 9} - \frac{1}{5! \cdot 11} + \cdots
\]
and computing the denominators,
\[
= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \cdots
\]
Since the first term whose absolute value is $< \frac{1}{100}$ is $\frac{1}{216}$, the correct estimate is
\[
1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42}
\]
(You did not need to put this into lowest terms: $\frac{26}{35}$.) This estimates the integral as $\approx 0.74286$ while the actual value is $\approx 0.74682$.

5. For each of the following, evaluate the series (that is, find its value), or state that it diverges. In each, simplify your answer as much as possible, and draw a box around your final answer.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^{n+1}}{n+1}$.

The series diverges by the ratio test or the divergence test. This series can be obtained by plugging $x = 5$ into $\ln(x + 1)$ but the power series converges only on the interval $(-1, 1]$, so the series above is not equal to $\ln(6)$. (It diverges.)

(b) $\sum_{n=0}^{\infty} \frac{(\ln 3)^{n+1}}{n!} = \ln 3 + (\ln 3)^2 + \frac{(\ln 3)^3}{2!} + \frac{(\ln 3)^4}{3!} + \frac{(\ln 3)^5}{4!} + \frac{(\ln 3)^6}{5!} + \cdots$

From the factorial denominators, we can guess that the series in play here is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, which converges for all real numbers. We have
\[
\sum_{n=0}^{\infty} \frac{(\ln 3)^{n+1}}{n!} = (\ln 3) \sum_{n=0}^{\infty} \frac{(\ln 3)^n}{n!} = (\ln 3) \cdot e^{\ln 3} = 3 \ln 3
\]
6. Consider the power series \( F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{4^n \cdot n^3} \).

(a) Show the algebraic work required to verify that \( F(2) = \sum_{n=1}^{\infty} \frac{2 \cdot (-1)^n}{n^3} \).

\[
F(2) = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^{2n+1}}{4^n \cdot n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^{2n} \cdot 2}{4^n \cdot n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 4^n \cdot 2}{4^n \cdot n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2}{n^3} = \sum_{n=1}^{\infty} \frac{2 \cdot (-1)^n}{n^3}
\]

(b) What value of \( L \) will you get if you apply the ratio test to the series in (a)?

\[ \text{You do not need to show any work.} \]

**NOTE:** “The series in (a)” is that for \( F(2) \), not \( F(x) \). We decided to give full credit to those of you who misinterpreted the question and performed the ratio test on \( F(x) \).

If you apply the ratio test to \( F(2) \) you will get \( L = 1 \). This is because the terms of the series are **algebraic**, and the ratio test typically only succeeds when the series has factorial or exponential terms \((-1)^n\) does not really count as exponential here, since it does not grow or decay). You can also see this by performing the Ratio Test as you usually do.

If you apply the ratio test to \( F(x) \) you will get \( L = \frac{1}{4} |x|^2 \) (note that plugging in \( x = 2 \) gives \( L = 1 \), as expected by the previous paragraph).

(c) Suppose \( \epsilon \) is a very small positive number \( (\epsilon \leq 10^{-10} \) for example). At which of these values will \( F(x) \) converge? **Circle all correct answers.**

i. \( x = 2 - \epsilon \)

The power series **CONVERGES** here. \( 2 \) is an endpoint of the interval of convergence (since \( L = 1 \) there), and the center of the power series is \( 0 \). Since \( 2 - \epsilon \) is between \( 0 \) and \( 2 \), the power series will converge at \( 2 - \epsilon \).

ii. \( x = 2 \)

The power series **CONVERGES** here, since the series \( F(2) \) (from part (a)) converges.

iii. \( x = 2 + \epsilon \)

The power series **DIVERGES** here, since \( 2 + \epsilon > 2 \), and \( 2 \) is an endpoint (the right/upper endpoint) of the IoC.
7. Find the endpoints of the interval of convergence of the power series

\[ \sum_{n=1}^{\infty} \frac{(2n)!(x - 1)^n}{8^n(n!)^2}. \]

Show all work, and draw a box around your final answer.

Using \( a_n = \frac{(2n)!(x - 1)^n}{8^n(n!)^2} \), we have \( a_{n+1} = \frac{(2n + 2)!(x - 1)^{n+1}}{8^{n+1}((n + 1)!)^2} \). Performing the ratio test,

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(2n + 2)!(x - 1)^{n+1}}{8^{n+1}((n + 1)!)^2} \cdot \frac{8^n(n!)^2}{(2n)!(x - 1)^n} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{(2n + 2)(2n + 1)(x - 1)}{8(n + 1)^2} \right|
\]

\[
= \frac{|x - 1|}{8} \lim_{n \to \infty} \left| \frac{4n^2 + 5n + 2}{n^2 + 2n + 1} \right|
\]

\[
= \frac{|x - 1|}{2}
\]

The endpoints are located where \( \frac{1}{2}|x - 1| = 1 \), so we need to solve \( \frac{1}{2}(x - 1) = -1 \) and \( \frac{1}{2}(x - 1) = +1 \) for \( x \), and these yield the endpoints \(-1\) and \(3\).
For the six series 8–13 below, fill in the required information:

(a) Whether the series converges or diverges;
(b) Which test(s) you ultimately used to determine convergence/divergence; and
(c) Additional info: If you used the integral test, write down the value of the relevant integral (or that it diverges); if you used a direct or limit comparison test, state what series you compared the given series to; if you used the ratio test, state the value of $L$ you found. (Leave blank for other tests.)

The next page can be used for any extra work, but it will not be graded.

### Series 8
\[ \sum_{n=0}^{\infty} \frac{n^2 \cdot 5^n}{(2n)!} \]

<table>
<thead>
<tr>
<th>(a) Converge/diverge?</th>
<th>(b) Test(s) used</th>
<th>(c) Additional info</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converges</td>
<td>Ratio test</td>
<td>$L = 0$</td>
</tr>
</tbody>
</table>

### Series 9
\[ \sum_{n=0}^{\infty} \frac{5n^2}{3^n \sqrt{n+1}} \]

<table>
<thead>
<tr>
<th>(a) Converge/diverge?</th>
<th>(b) Test(s) used</th>
<th>(c) Additional info</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converges</td>
<td>Ratio test</td>
<td>$L = 1/3$</td>
</tr>
</tbody>
</table>

### Series 10
\[ \sum_{n=1}^{\infty} \frac{n^5 + n \ln n}{\sqrt{n^{13} + 5n^7 + 6n^3 + 2n + 1}} \]

<table>
<thead>
<tr>
<th>(a) Converge/diverge?</th>
<th>(b) Test(s) used</th>
<th>(c) Additional info</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converges</td>
<td>Limit comparison</td>
<td>With the $p$-series $\sum \frac{n^\frac{5}{2}}{\sqrt{n^{13}}} = \sum \frac{1}{n^{1/2}}$.</td>
</tr>
</tbody>
</table>

### Series 11
\[ \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(2 - \frac{1}{n})} \]

<table>
<thead>
<tr>
<th>(a) Converge/diverge?</th>
<th>(b) Test(s) used</th>
<th>(c) Additional info</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diverges</td>
<td>Divergence test</td>
<td>Since $\lim_{n \to \infty} \ln(2 - \frac{1}{n}) = \ln 2$</td>
</tr>
</tbody>
</table>
12. $\sum_{n=1}^{\infty} \frac{\cos n}{n^{3/2} + 1}$

<table>
<thead>
<tr>
<th>(a) Converge/diverge?</th>
<th>(b) Test(s) used</th>
<th>(c) Additional info</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converges</td>
<td>Absolute convergence test and then direct comparison</td>
<td>With $\sum \frac{1}{n^{3/2}}$</td>
</tr>
</tbody>
</table>

13. $\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{\ln(n)}$

<table>
<thead>
<tr>
<th>(a) Converge/diverge?</th>
<th>(b) Test(s) used</th>
<th>(c) Additional info</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converges</td>
<td>Alternating series test</td>
<td>Since $\cos(\pi n) = (-1)^n$</td>
</tr>
</tbody>
</table>

The remaining space is provided for any extra work. If you think this work is important to one of your solutions, please indicate that on the page of the relevant problem (otherwise we won’t know to look!).
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